Presentations for singular wreath products

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Abstract

For a monoid M and a subsemigroup S of the full transformation semigroup $\mathcal{T}_n$, the wreath product $M \wr S$ is defined to be the semidirect product $M^n \rtimes S$, with the coordinatewise action of S on M^n. The full wreath product $M \wr \mathcal{T}_n$ is isomorphic to the endomorphism monoid of the free M-act on n generators. Here, we are particularly interested in the case that $S = \mathrm{Sing}_n$ is the singular part of $\mathcal{T}_n$, consisting of all non-invertible transformations. Our main results are presentations for $M \wr \mathrm{Sing}_n$ in terms of certain natural generating sets, and we prove these via general results on semidirect products and wreath products. We re-prove a classical result of Bulman-Fleming that $M \wr \mathrm{Sing}_n$ is idempotent generated if and only if the set $M/\mathscr{L}$ of $\mathscr{L}$-classes of M forms a chain under the usual ordering of $\mathscr{L}$-classes, and we give a presentation for $M \wr \mathrm{Sing}_n$ in terms of idempotent generators for such a monoid M. Among other results, we also give estimates for the minimal size of a generating set for $M \wr \mathrm{Sing}_n$, as well as exact values in some cases (including the case that M is finite and $M/\mathscr{L}$ is a chain, in which case we also calculate the minimal size of an idempotent generating set). As an application of our results, we obtain a presentation (with idempotent generators) for the idempotent generated subsemigroup of the endomorphism monoid of a uniform partition of a finite set.

Keywords: Wreath products, semidirect products, transformation semigroups, presentations, rank, idempotent rank.

MSC: 20M05; 20M20.

1 Introduction

We draw our main inspiration from three closely related themes in semigroup theory: idempotent generation, endomorphism monoids, and presentations. Idempotents have long played an important role in algebraic and combinatorial semigroup theory. On the one hand, Howie's 1966 article [49] on singular transformation semigroups initiated a vast research theme in idempotent generated semigroups, with much attention focused on (partial) endomorphism monoids of sets, vector spaces, modules, independence algebras, free acts and other structures [3,6,7,9,10,15,25,29–31,35–38,41,43,44,50,52,57,60,71,75,76]. Among other things, Howie's article [49] demonstrated a universal property: every semigroup S embeds in an idempotent generated (singular transformation) semigroup that may be taken to be finite if S is finite. On the other hand, every idempotent generated semigroup T is a homomorphic image of a so-called *free idempotent generated semigroup* that has the same *biordered set* of idempotents as T. These semigroups (henceforth referred to as FIGSs) are defined by means of a presentation (by generators and relations) derived from abstract biordered sets [22, 65]. Among other things, these FIGSs encode combinatorial and topological properties of semigroups, as their maximal subgroups are isomorphic to the fundamental groups of certain complexes arising from the biordered set [8]. These maximal subgroups have therefore been the subject of intense study, particularly in the last decade or so. Early results [63, 66, 67] led to a conjecture that these maximal subgroups were always free, but this turned out to be false. The first counter-example was provided in [8] and, soon after, it was shown by Gray and Ruškuc [46] that *every* group occurs as a maximal subgroup of some FIGS. A recent focus has, therefore, been to describe the maximal subgroups of FIGSs arising from

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biordered sets of well-known semigroups [11,13,14,16,19,45,62], while relatively fewer studies of the global structure of a FIGS have been carried out [12,54]. The Gray-Ruškuc result [46] has now been proved in a number of different ways [11,20,42], with endomorphism monoids of free G-acts playing a key role in some of these later proofs, and providing a natural biordered set for the construction. It has long been known that the endomorphism monoid of a free G-act of (finite) rank n is isomorphic to a wreath product $G \wr \mathcal{T}_n$; in fact, this is true in the more general case of free M-acts, where M is a monoid [55,77]. It is therefore very natural to study the structure of wreath products $M \wr \mathcal{T}_n$ or, more generally, $M \wr S$ for an arbitrary subsemigroup S of $\mathcal{T}_n$. Further motivation for studying such wreath products comes from the fact that the wreath product $\mathcal{T}_m \wr \mathcal{T}_n$ of two transformation semigroups is isomorphic to the endomorphism monoid of a uniform partition with n blocks of size m [5,17,68].

Close to the time that Howie's article [49] appeared, a number of authors obtained presentations for various monoids of (partial) transformations [1,2,69,70], sparking a field of research that continues to this day; see for example [23,24,32,56,58], and especially the survey [33] and references therein. Presentations have recently been obtained for certain singular semigroups of transformations and related structures [25–28,61]. In particular, a presentation for Sing_n, the singular part of $\mathcal{T}_n$, is given in [26] in terms of the generating set consisting of all idempotents of rank $n-1$. To the authors' knowledge, a general presentation for the endomorphism monoid $\mathrm{End}(A)$ of an arbitrary independence algebra A is not currently known. But for a special subclass of such algebras, the above-mentioned free G-acts of finite rank, such a presentation can be described using results of Lavers [59] on general products of *monoids*, since (as noted above) these endomorphism monoids are isomorphic to wreath products of the form $G \wr \mathcal{T}_n$. Here, we are interested in the more general problem of finding presentations for wreath products $M \wr S$ for an arbitrary monoid M and an arbitrary *subsemigroup* $S \subseteq \mathcal{T}_n$, particularly in the case that $S = \mathrm{Sing}_n$. This kind of problem can be quite difficult in the case that S does not contain the identity transformation (as happens when $S = \mathrm{Sing}_n$, for example), since many articles on presentations for semigroup constructions (including wreath and semidirect products) focus on the case of monoids [34,53,59,74]; notable exceptions that are not restricted to monoids have concentrated on constructions that do not capture the kind of wreath and semidirect products that arise from endomorphisms of M-acts [21,73]. As such, to achieve our main aim of finding presentations for wreath products $M \wr \mathrm{Sing}_n$, we first prove general results on presentations for arbitrary semidirect products $M \rtimes S$ (of which the wreath product is a special case) where M is a monoid and S a semigroup. From these, we are able to deduce a number of presentations for $M \wr \mathrm{Sing}_n$ that extend the presentation for Sing_n from [26]. Along the way, we obtain several other results of independent interest, as we describe below.

The article is organised as follows. In Section 2, we establish notation and gather some background results on (transformation) semigroups and presentations. In Section 3, we give a general presentation for a semidirect product $M \rtimes S$ of a monoid M and semigroup S (Theorem 3.1), including wreath products as a special case. In Section 4, we classify and enumerate the idempotents of $M \wr \mathcal{T}_n$ (Proposition 4.3 and Corollary 4.5), describe canonical generating sets for $M \wr \mathrm{Sing}_n$ (Theorem 4.7), give necessary and sufficient conditions for $M \wr \mathrm{Sing}_n$ to be idempotent generated (also contained in Theorem 4.7, and originally proved by Bulman-Fleming [9]), and give bounds (and exact values in the idempotent generated case) for the minimal size of a generating set for finite $M \wr \mathrm{Sing}_n$ (Proposition 4.10 and Theorem 4.12; see also Theorem 4.14). In Section 5, we give a number of presentations for $M \wr \mathrm{Sing}_n$ with respect to the canonical generating sets alluded to above (Corollary 5.1 and Theorem 5.2); in the case that $M \wr \mathrm{Sing}_n$ is idempotent generated, we give a presentation in terms of a particularly natural idempotent generating set (Theorems 5.9 and 5.12). Finally, in Section 6, we apply our results to obtain presentations for the idempotent generated subsemigroups of a class of monoids including the endomorphism monoid of a uniform partition of a finite set (see Theorem 6.3).

2 Preliminaries

Let S be a semigroup, and write S^1 for the monoid obtained by adjoining an identity 1 to S, if necessary. Unless otherwise specified, we will generally write 1 for the identity element of any monoid. Recall that *Green's preorders* are defined, for $a, b \in S$, by

$$a \leq_{\mathscr{R}} b \ \Leftrightarrow \ a \in bS^1, \quad a \leq_{\mathscr{L}} b \ \Leftrightarrow \ a \in S^1 b, \quad a \leq_{\mathscr{J}} b \ \Leftrightarrow \ a \in S^1 b S^1,$$

and that *Green's relations* are defined by

$$\mathscr{R} = {\leq_{\mathscr{R}}} \cap {\geq_{\mathscr{R}}}, \quad \mathscr{L} = {\leq_{\mathscr{L}}} \cap {\geq_{\mathscr{L}}}, \quad \mathscr{J} = {\leq_{\mathscr{J}}} \cap {\geq_{\mathscr{J}}}, \quad \mathscr{H} = \mathscr{R} \cap \mathscr{L}, \quad \mathscr{D} = \mathscr{R} \circ \mathscr{L}.$$

It can easily be proved that $\mathscr{D} = \mathscr{R} \circ \mathscr{L} = \mathscr{L} \circ \mathscr{R}$ (so that $\mathscr{D}$ is the *least upper bound* of $\mathscr{R}$ and $\mathscr{L}$ in the lattice of equivalence relations on S). If S is finite, then $\mathscr{D} = \mathscr{J}$. If $a \in S$, and $\mathscr{K}$ is one of Green's relations, we write K_a for the $\mathscr{K}$-class of S containing a. If a subset $A \subseteq S$ is a union of $\mathscr{K}$-classes, we write $A/\mathscr{K} = \{K_a : a \in A\}$ for the set of all $\mathscr{K}$-classes contained in A. If $\mathscr{K}$ is one of $\mathscr{R}, \mathscr{L}, \mathscr{J}$, then $\leq_{\mathscr{K}}$ induces a partial order on $S/\mathscr{K}$; as usual, we will generally write $\leq_{\mathscr{K}}$ for this induced order, so that $a \leq_{\mathscr{K}} b$ in $S \Leftrightarrow K_a \leq_{\mathscr{K}} K_b$ in $S/\mathscr{K}$. For any subset $A \subseteq S$, we write $E(A) = \{a \in A : a = a^2\}$ for the set of idempotents in A, and $\langle A \rangle$ for the subsemigroup of S generated by A. The *rank* of a semigroup S, denoted $\operatorname{rank}(S)$, is the smallest size of a generating set for S. If S is idempotent generated, then the *idempotent rank*, denoted $\operatorname{idrank}(S)$, is defined analogously with respect to generating sets consisting of idempotents. For more background on semigroups, see [48, 51].

Let X be an *alphabet* (a set whose elements are called *letters*), and denote by X^+ (resp., X^*) the free semigroup (resp., free monoid) on X. We denote the *empty word* (over any alphabet) by 1, so $X^* = X^+ \cup \{1\}$. If $R \subseteq X^+ \times X^+$ (resp., $R \subseteq X^* \times X^*$), we denote by $R^\sharp$ the congruence on X^+ (resp., X^*) generated by R. We say a semigroup (resp., monoid) S has *semigroup* (resp., *monoid*) *presentation* $\langle X : R \rangle$ if $S \cong X^+/R^\sharp$ (resp., $S \cong X^*/R^\sharp$) or, equivalently, if there is an epimorphism $X^+ \to S$ (resp., $X^* \to S$) with kernel $R^\sharp$. If ϕ is such an epimorphism, we say S has *presentation* $\langle X : R \rangle$ *via* ϕ. The elements of R are generally referred to as *relations*, and a relation $(w_1, w_2) \in R$ will usually be displayed as an equation: $w_1 = w_2$. Unless otherwise specified (and this will only occur in Section 6), all presentations will be semigroup presentations.

For an integer $n \geq 0$, we write $\mathbf{n} = \{1, \ldots, n\}$ and $\mathcal{T}_n$ for the *full transformation semigroup of degree n*, which consists of all transformations of $\mathbf{n}$ (i.e., all maps $\mathbf{n} \to \mathbf{n}$) under composition. (When $n = 0$, $\mathbf{n} = \emptyset$ and $\mathcal{T}_0$ consists only of the empty function $\emptyset \to \emptyset$.) For $\alpha \in \mathcal{T}_n$ and $i \in \mathbf{n}$, we write $i\alpha$ for the image of i under α, so that transformations are composed left-to-right. For $\alpha \in \mathcal{T}_n$, define

$$\operatorname{im}(\alpha) = \{i\alpha : i \in \mathbf{n}\}, \quad \ker(\alpha) = \{(i,j) \in \mathbf{n} \times \mathbf{n} : i\alpha = j\alpha\}, \quad \operatorname{rank}(\alpha) = |\operatorname{im}(\alpha)|.$$

It is well known (see [51, Exercise 16]) that, for $\alpha, \beta \in \mathcal{T}_n$,

$$\alpha \leq_{\mathscr{R}} \beta \;\Leftrightarrow\; \ker(\alpha) \supseteq \ker(\beta), \quad \alpha \leq_{\mathscr{L}} \beta \;\Leftrightarrow\; \operatorname{im}(\alpha) \subseteq \operatorname{im}(\beta), \quad \alpha \leq_{\mathscr{J}} \beta \;\Leftrightarrow\; \operatorname{rank}(\alpha) \leq \operatorname{rank}(\beta).$$

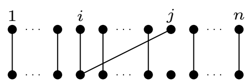
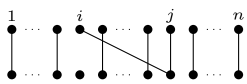
Since $\mathcal{T}_n$ is finite (of size n^n), $\mathscr{D} = \mathscr{J}$. The next result is well known, and is easily proved.

Proposition 2.1. *A transformation $\alpha \in \mathcal{T}_n$ is an idempotent if and only if the restriction $\alpha|_{\operatorname{im}(\alpha)}$ of α to $\operatorname{im}(\alpha)$ is the identity map. Consequently, $|E(\mathcal{T}_n)| = \sum_{k=1}^n \binom{n}{k} k^{n-k}$.* □

The group of units of $\mathcal{T}_n$ is the *symmetric group* $\mathcal{S}_n = \{\alpha \in \mathcal{T}_n : \operatorname{rank}(\alpha) = n\}$. We write $\operatorname{Sing}_n = \mathcal{T}_n \setminus \mathcal{S}_n$ for the *singular part* of $\mathcal{T}_n$, which consists of all non-invertible (i.e., *singular*) transformations of $\mathbf{n}$. A famous result of Howie [49] states that Sing_n is generated by its idempotents: in fact, by its idempotents of rank $n-1$. The latter are precisely the maps ε_{ij} (for $i, j \in \mathbf{n}$ with $i \neq j$) defined by

$$k\varepsilon_{ij} = \begin{cases} k & \text{if } k \neq j \\ i & \text{if } k = j. \end{cases}$$

As usual, these idempotents may be represented diagrammatically, for $1 \leq i < j \leq n$, by

$\varepsilon_{ij} =$  and $\varepsilon_{ji} =$  .

We will write $\mathcal{X} = \{\varepsilon_{ij} : i, j \in \mathbf{n}, \ i \neq j\}$ for the set of all rank $n-1$ idempotents from $\mathcal{T}_n$. Note that $|\mathcal{X}| = 2\binom{n}{2} = n(n-1)$. It is easy to check (and follows from facts mentioned above) that for all $i, j, k, l \in \mathbf{n}$ with $i \neq j$ and $k \neq l$,

$$\varepsilon_{ij} \mathscr{R} \varepsilon_{kl} \;\Leftrightarrow\; \{i,j\} = \{k,l\} \qquad \text{and} \qquad \varepsilon_{ij} \mathscr{L} \varepsilon_{kl} \;\Leftrightarrow\; j = l.$$

The next result is [49, Theorem I].

Theorem 2.2. *If $n \geq 2$, then* $\mathrm{Sing}_n = \langle \mathcal{X} \rangle$. □

Note that $\mathrm{Sing}_n = \emptyset$ if $n \leq 1$. Note also that $\mathrm{Sing}_2 = \mathcal{X} = \{\varepsilon_{12}, \varepsilon_{21}\}$ is a right-zero semigroup, so the smallest (idempotent) generating set for Sing_2 has size $2 = |\mathcal{X}|$. For $n \geq 3$, Sing_n may be generated by $\frac{1}{2}|\mathcal{X}|$ elements, by further results of Howie, as we now explain. For a subset $F \subseteq \mathcal{X}$, let Γ_F be the directed graph with vertex set $\mathbf{n}$ and edge set $\{(i,j) : \varepsilon_{ij} \in F\}$. We say a directed graph Γ is *complete* if its associated undirected graph (obtained by changing directed edges to undirected edges, and converting any resulting parallel edges to single edges) is the complete graph (with the same vertex set as Γ). Part (i) of the next result is [41, Theorem 2.1], and Part (ii) is [50, Theorem 1]. Recall that a directed graph is *strongly connected* if, for any pair of vertices x, y, there is a (possibly empty) directed path from x to y.

Theorem 2.3. *Let $n \geq 3$. Then*

(i) $\mathrm{rank}(\mathrm{Sing}_n) = \mathrm{idrank}(\mathrm{Sing}_n) = \binom{n}{2} = \frac{1}{2}n(n-1)$;

(ii) *if $F \subseteq \mathcal{X}$, then $\mathrm{Sing}_n = \langle F \rangle$ if and only if Γ_F is strongly connected and complete.* □

A presentation for Sing_n was given in [26], in terms of the generating set $\mathcal{X}$. Define an alphabet

$$X = \{e_{ij} : i, j \in \mathbf{n},\ i \neq j\},$$

an epimorphism

$$\phi : X^+ \to \mathrm{Sing}_n : e_{ij} \mapsto \varepsilon_{ij},$$

and let R be the set of relations

$$\begin{aligned}
e_{ij}^2 = e_{ij} &= e_{ji}e_{ij} && \text{for distinct } i,j && \text{(R1)}\\
e_{ij}e_{kl} &= e_{kl}e_{ij} && \text{for distinct } i,j,k,l && \text{(R2)}\\
e_{ik}e_{jk} &= e_{ik} && \text{for distinct } i,j,k && \text{(R3)}\\
e_{ij}e_{ik} = e_{ik}e_{ij} &= e_{jk}e_{ij} && \text{for distinct } i,j,k && \text{(R4)}\\
e_{ki}e_{ij}e_{jk} &= e_{ik}e_{kj}e_{ji}e_{ik} && \text{for distinct } i,j,k && \text{(R5)}\\
e_{ki}e_{ij}e_{jk}e_{kl} &= e_{ik}e_{kl}e_{li}e_{ij}e_{jl} && \text{for distinct } i,j,k,l. && \text{(R6)}
\end{aligned}$$

The next result is [26, Theorem 6].

Theorem 2.4. *For $n \geq 2$, the semigroup Sing_n has presentation $\langle X : R \rangle$ via ϕ.* □

3 Semidirect products and wreath products

In this section, we prove some general results about semidirect products $M \rtimes S$ and wreath products $M \wr S$, where M is a monoid and S a semigroup (S will be a subsemigroup of some $\mathcal{T}_n$ in the case of $M \wr S$).

3.1 Semidirect products

Let S be a semigroup and M a monoid with identity 1. (Note that S might also be a monoid.) Suppose S has a left action on M by monoid endomorphisms; that is, there is a homomorphism $\varphi : S \to \mathrm{End}^*(M) : s \mapsto \varphi_s$, where $\mathrm{End}^*(M)$ denotes the monoid of endomorphisms of M with right-to-left composition. For $s \in S$ and $a \in M$, we write $s \cdot a = \varphi_s(a)$. So

$$s \cdot 1 = 1, \quad s \cdot (t \cdot a) = (st) \cdot a, \quad s \cdot (ab) = (s \cdot a)(s \cdot b) \qquad \text{for all } s,t \in S \text{ and } a,b \in M.$$

(Note that if S happens to be a monoid, we do not assume that it acts *monoidally* on M: i.e., we do not assume that the identity of S acts as the identity automorphism of M.) The *semidirect product* $M \rtimes S = M \rtimes_\varphi S$ has underlying set $M \times S = \{(a,s) : a \in M,\ s \in S\}$, and product defined by

$$(a,s)(b,t) = (a(s \cdot b), st) \qquad \text{for all } s,t \in S \text{ and } a,b \in M.$$

The fact that S acts by *monoid* endomorphisms ensures that S may be identified with the subsemigroup $\{(1,s) : s \in S\}$ of $M \rtimes S$. If S is a monoid acting monoidally on M (i.e., $1 \cdot a = a$ for all $a \in M$), then $\{(a,1) : a \in M\}$ is an isomorphic copy of M inside $M \rtimes S$. However, this article is mostly concerned with the case that S is not a monoid, in which case $M \rtimes S$ does not contain such a canonical copy of M. A motivating example of the semidirect product are the *wreath products*, which are the subject of Section 3.2.

Suppose now that S has semigroup presentation $\langle X : R \rangle$ via $\phi : X^+ \to S$. Define a new alphabet $X_M = \{x_a : x \in X,\ a \in M\}$. We regard X as a subset of X_M by identifying $x \in X$ with $x_1 \in X_M$. For a word $w = x_1 \cdots x_k \in X^+$, and for an element $a \in M$, we define the word $w_a = (x_1)_a x_2 \cdots x_k \in X_M^+$. Consider the set $R_M = R_M^1 \cup R_M^2$ of relations over X_M, where R_M^1 and R_M^2 are defined by

$$R_M^1 = \{(u_a, v_a) : (u,v) \in R,\ a \in M\} \qquad \text{and} \qquad R_M^2 = \{(x_a y_b, x_{a(x\phi \cdot b)} y) : x, y \in X,\ a, b \in M\}.$$

Note that, by identifying $X \subseteq X_M$ as above, we also have $R \subseteq R_M$, via $(u,v) \equiv (u_1, v_1)$. Define a map

$$\phi_M : X_M^+ \to M \rtimes S \qquad \text{by} \qquad x_a \phi_M = (a, x\phi) \qquad \text{for all } x \in X \text{ and } a \in M.$$

It is easy to check that $w_a \phi_M = (a, w\phi)$ for all $a \in M$ and $w \in X^+$. It quickly follows (from the surjectivity of $\phi : X^+ \to S$) that ϕ_M is surjective. For convenience, in the following proof, even though R and R_M may not be symmetric, we will say "$(u,v) \in R$" to mean "$(u,v) \in R$ or $(v,u) \in R$" (and similarly for R_M).

Theorem 3.1. *With the above notation, $M \rtimes S$ has semigroup presentation $\langle X_M : R_M \rangle$ via ϕ_M.*

Proof. We showed above that ϕ_M is surjective, so we just need to show that $\ker(\phi_M) = R_M^\sharp$. First note that for any $(u,v) \in R$ and $a \in M$, $u_a \phi_M = (a, u\phi) = (a, v\phi) = v_a \phi_M$, while for any $x, y \in X$ and $a, b \in M$, $(x_a y_b)\phi_M = (a, x\phi)(b, y\phi) = (a(x\phi \cdot b), (xy)\phi) = (a(x\phi \cdot b), x\phi)(1, y\phi) = (x_{a(x\phi \cdot b)} y)\phi_M$, showing that $R_M \subseteq \ker(\phi_M)$.

Conversely, suppose $u = (x_1)_{a_1} \cdots (x_k)_{a_k}, v = (y_1)_{b_1} \cdots (y_l)_{b_l} \in X_M^+$ are such that $u\phi_M = v\phi_M$. For the remainder of the proof, write $\approx = R_M^\sharp$. Using relations from R_M^2, we have

$$u \approx (x_1)_a x_2 \cdots x_k = (x_1 \cdots x_k)_a \qquad \text{and} \qquad v \approx (y_1)_b y_2 \cdots y_l = (y_1 \cdots y_l)_b \qquad \text{for some } a, b \in M.$$

Since $\approx \subseteq \ker(\phi_M)$, we have

$$(a, (x_1 \cdots x_k)\phi) = (x_1 \cdots x_k)_a \phi_M = u\phi_M = v\phi_M = (y_1 \cdots y_l)_b \phi_M = (b, (y_1 \cdots y_l)\phi).$$

It follows that $a = b$ and $(x_1 \cdots x_k)\phi = (y_1 \cdots y_l)\phi$. Since $\ker(\phi) = R^\sharp$, it follows that there is a sequence of words $x_1 \cdots x_k = w_0, w_1, \ldots, w_r = y_1 \cdots y_l$ such that, for each $0 \le i \le r-1$, $w_i = w_i' u w_i''$ and $w_{i+1} = w_i' v w_i''$ for some $w_i', w_i'' \in X^*$ and $(u,v) \in R$. But then we see that $(w_i)_a \approx (w_{i+1})_a$, using either $(u,v) \in R \subseteq R_M$ (if w_i' is non-empty) or $(u_a, v_a) \in R_M$ (if w_i' is empty). But then

$$u \approx (x_1 \cdots x_k)_a = (w_0)_a \approx (w_1)_a \approx \cdots \approx (w_r)_a = (y_1 \cdots y_l)_a = (y_1 \cdots y_l)_b \approx v,$$

completing the proof. □

The next result follows immediately from Theorem 3.1

Corollary 3.2. *If S is finitely presented and M is finite, then $M \rtimes S$ is finitely presented.* □

Of course the converse of Corollary 3.2 is not true: for example, the semidirect product $M \rtimes S$ of finitely presented infinite *monoids* (with a monoidal action of S on M) is finitely presented [59, Corollary 2]. One might hope to improve Corollary 3.2 by assuming only that M is finitely presented (rather than the stronger assumption of being finite). But this is not the case in general, as the following examples show.

Example 3.3. Let $\mathbb{N}$ be the additive monoid of natural numbers (including 0), and let $S = \{x, x^2\}$ with $x^3 = x^2 \neq x$. It is easy to show that any generating set for the *direct* product $\mathbb{N} \times S$ must contain $\mathbb{N} \times \{x\}$. In particular, $\mathbb{N} \times S$ is not finitely generated (and, hence, not finitely presented).

Example 3.4. Our second example is a semidirect product that is not direct. Let $M = \mathbb{N} \times \mathbb{N}$, and let $S = \{\varepsilon\}$ be a trivial semigroup. Define an action of S on M by $\varepsilon \cdot (a,b) = (a,a)$. Since $|S| = 1$, we may identify the element $((a,b), \varepsilon)$ of $M \rtimes S$ with (a,b): in this way, the operation on $M \rtimes S$ obeys $(a,b)(c,d) = (a+c, b+c)$. It is again easy to show that any generating set for $M \rtimes S$ must contain $\mathbb{N} \times \{0\}$. (In fact, this example can be viewed as a *wreath product*; see Section 3.2 for more details.)

3.2 Wreath products

Let S be a subsemigroup of the full transformation semigroup $\mathcal{T}_n$, and let M be an arbitrary monoid. Then S has a natural left action on M^n (the direct product of n copies of M) given by

$$\alpha \cdot (a_1, \ldots, a_n) = (a_{1\alpha}, \ldots, a_{n\alpha}) \qquad \text{for } \alpha \in S \text{ and } a_1, \ldots, a_n \in M.$$

The resulting semidirect product $M^n \rtimes S$ is the *wreath product* of M by S, denoted $M \wr S$. Multiplication in $M \wr S$ obeys the rule

$$((a_1, \ldots, a_n)\alpha)((b_1, \ldots, b_n), \beta) = ((a_1 b_{1\alpha}, \ldots, a_n b_{n\alpha}), \alpha\beta).$$

For example, writing $\varepsilon = \varepsilon_{12} \in \mathcal{T}_2$, if $M = \mathbb{N}$ and $S = \{\varepsilon\}$, then $M \wr S$ is the semigroup from Example 3.4. When $S = \mathcal{T}_n$, we obtain the *full* wreath product $M \wr \mathcal{T}_n$. When $S = \mathrm{Sing}_n = \mathcal{T}_n \setminus \mathcal{S}_n$, we obtain the *singular* wreath product $M \wr \mathrm{Sing}_n$; these singular wreath products are the main focus of this article. If $M = \{1\}$, then $M \wr S \cong S$ for any S. On the other hand, if $S = \{1\}$, where $1 \in \mathcal{T}_n$ denotes the identity map, then $M \wr S \cong M^n$.

There is a useful way to picture an element $(\mathbf{a}, \alpha)$ of $M \wr \mathcal{T}_n$. We draw two parallel rows of vertices, both labelled $1, \ldots, n$ (and assumed to be increasing from left to right, unless otherwise specified); for each $i \in \mathbf{n}$, we draw an edge between upper vertex i and lower vertex $i\alpha$; and we decorate upper vertex i with the monoid element a_i, where $\mathbf{a} = (a_1, \ldots, a_n)$. For example, two elements $(\mathbf{a}, \alpha), (\mathbf{b}, \beta) \in M \wr \mathcal{T}_6$ are pictured in Figure 1. To calculate the product $(\mathbf{a}, \alpha)(\mathbf{b}, \beta) = (\mathbf{a}(\alpha \cdot \mathbf{b}), \alpha\beta)$ diagrammatically: we first stack $(\mathbf{a}, \alpha)$ above $(\mathbf{b}, \beta)$, identifying lower vertex i of $(\mathbf{a}, \alpha)$ with upper vertex i of $(\mathbf{b}, \beta)$ for each i; we then "slide" the b_i decorations up the edges of $(\mathbf{a}, \alpha)$ and form the appropriate products $a_j b_i$; finally, we straighten the remaining edges, and remove incomplete edges. An example calculation is also given in Figure 1. By convention, we will often omit the decoration on upper vertex i of $(\mathbf{a}, \alpha)$ if $a_i = 1$.

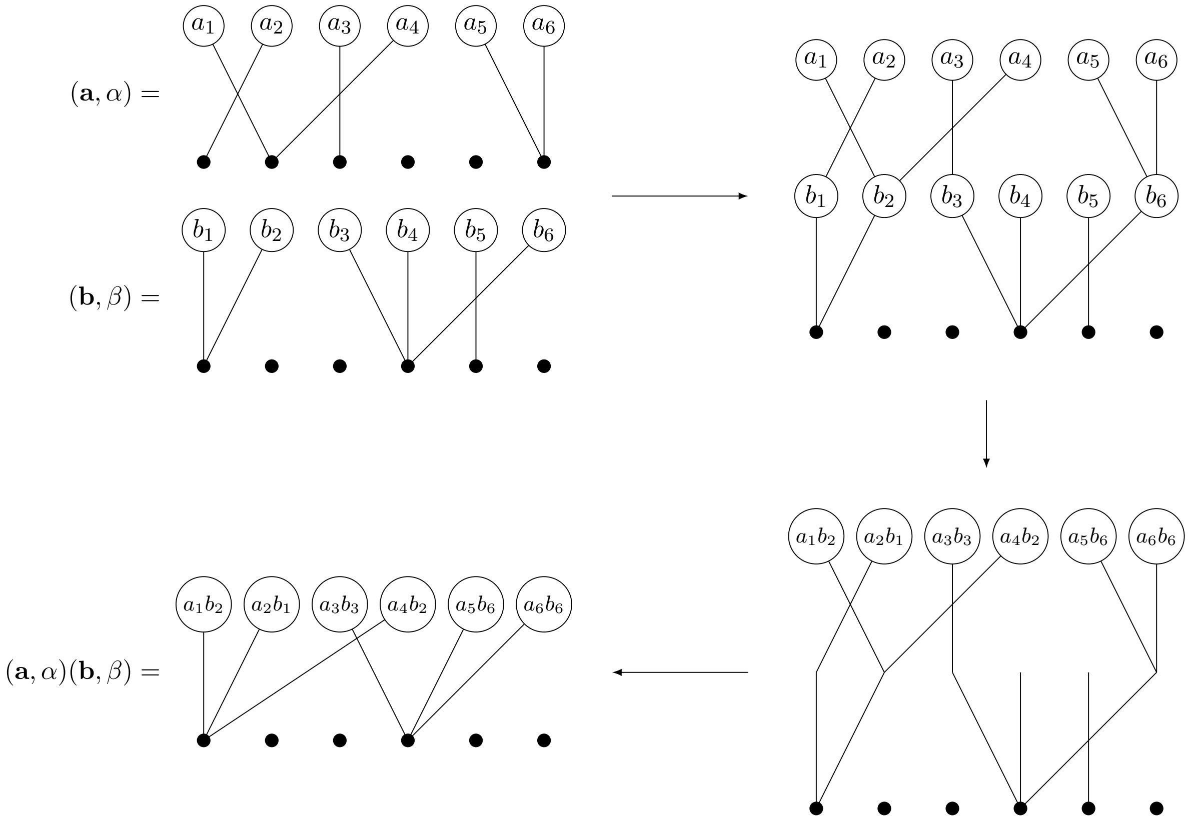


Figure 1: Elements of $M \wr \mathcal{T}_6$ (top left) and their product (bottom left).

Since $M \wr S = M^n \rtimes S$ is a semidirect product, Theorem 3.1 directly leads to a general presentation for $M \wr S$, modulo a presentation $\langle X : R \rangle$ for S, but we will not state this explicitly.

The remainder of the article almost exclusively concerns singular wreath products $M \wr \mathrm{Sing}_n$. Because Sing_n is empty for $n \leq 1$, we will assume that $n \geq 2$ whenever we make a statement about Sing_n.

4 Idempotents and generators for $M \wr \mathrm{Sing}_n$

In this section, we extend Proposition 2.1 to full wreath products $M \wr \mathcal{T}_n$ and Theorems 2.2 and 2.3 to singular wreath products $M \wr \mathrm{Sing}_n$. In particular, we characterise and enumerate the idempotents of $M \wr \mathcal{T}_n$ (Lemma 4.1, Proposition 4.3 and Corollary 4.5); we give canonical generators and re-prove Bulman-Fleming's [9] necessary and sufficient conditions for $M \wr \mathrm{Sing}_n$ to be idempotent generated (Theorem 4.7); and we give bounds (and exact values in some cases) on the minimal size of (idempotent) generating sets for $M \wr \mathrm{Sing}_n$ (Proposition 4.10 and Theorems 4.12 and 4.14).

For $i, j \in \mathbf{n}$ with $i \neq j$, and for $\mathbf{a} \in M^n$, we define $\varepsilon_{ij;\mathbf{a}} = (\mathbf{a}, \varepsilon_{ij}) \in M \wr \mathrm{Sing}_n$. As a special case, for $a, b \in M$, we define

$$\varepsilon_{ij;ab} = \varepsilon_{ij;\mathbf{a}} \qquad \text{where } \mathbf{a} \in M^n \text{ is defined by } a_k = \begin{cases} a & \text{if } k = i \\ b & \text{if } k = j \\ 1 & \text{otherwise.} \end{cases}$$

As a special case of the latter, we define $\varepsilon_{ij;a} = \varepsilon_{ij;1a}$, for $a \in M$. We gather these elements into the sets

$$\begin{aligned} \mathcal{X}_n &= \{\varepsilon_{ij;\mathbf{a}} : i, j \in \mathbf{n},\ i \neq j,\ \mathbf{a} \in M^n\}, \\ \mathcal{X}_2 &= \{\varepsilon_{ij;ab} : i, j \in \mathbf{n},\ i \neq j,\ a, b \in M\}, \\ \mathcal{X}_1 &= \{\varepsilon_{ij;a} : i, j \in \mathbf{n},\ i \neq j,\ a \in M\}. \end{aligned}$$

As usual, we also identify $\varepsilon_{ij} \in \mathrm{Sing}_n$ with $\varepsilon_{ij;1} \in M \wr \mathrm{Sing}_n$. So we have $\mathcal{X} \subseteq \mathcal{X}_1 \subseteq \mathcal{X}_2 \subseteq \mathcal{X}_n$. Note that

$$|\mathcal{X}| = 2\binom{n}{2}, \quad |\mathcal{X}_1| = 2|M| \times \binom{n}{2}, \quad |\mathcal{X}_2| = 2|M|^2 \times \binom{n}{2}, \quad |\mathcal{X}_n| = 2|M|^n \times \binom{n}{2}.$$

Note also that $\mathcal{X}_1 \subseteq E(M \wr \mathrm{Sing}_n)$, as we show in Figure 2 (and also follows from Lemma 4.1 below), but that elements of $\mathcal{X}_2$ need not be idempotent in general.

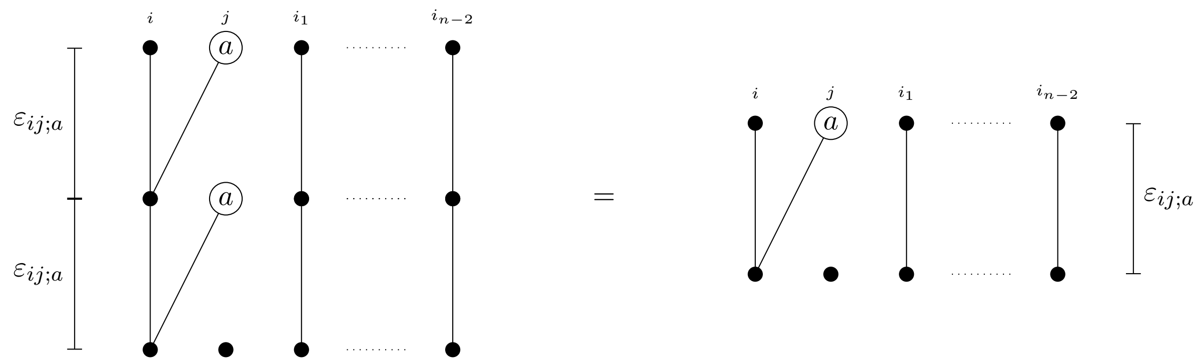


Figure 2: Diagrammatic proof that $\varepsilon_{ij;a}$ is an idempotent. Vertices are drawn in the order $i, j, i_1, \ldots, i_{n-2}$, where $\mathbf{n} \setminus \{i, j\} = \{i_1, \ldots, i_{n-2}\}$.

4.1 Idempotents in $M \wr \mathcal{T}_n$

We begin with a simple description of the idempotents of $M \wr \mathcal{T}_n$.

Lemma 4.1. *We have* $E(M \wr \mathcal{T}_n) = \{(\mathbf{a}, \alpha) \in M \wr \mathcal{T}_n : \alpha \in E(\mathcal{T}_n)$ *and* $a_i a_{i\alpha} = a_i$ *for all* $i \in \mathbf{n}\}$.

Proof. This follows immediately from the fact that $(\mathbf{a}, \alpha)^2 = ((a_1 a_{1\alpha}, \ldots, a_n a_{n\alpha}), \alpha^2)$. $\square$

Remark 4.2. If $(\mathbf{a}, \alpha) \in E(M \wr \mathcal{T}_n)$, then $a_i \in E(M)$ for all $i \in \mathrm{im}(\alpha)$, since $\alpha \in E(\mathcal{T}_n)$ acts as the identity on $\mathrm{im}(\alpha)$.

If M is infinite, then $|E(M \wr \mathcal{T}_n)| = |M| = |M \wr \mathcal{T}_n|$, since $|\mathcal{X}_1| = |M|$ and $\mathcal{X}_1 \subseteq E(M \wr \mathcal{T}_n)$. The next result calculates the number of idempotents in $M \wr \mathcal{T}_n$ when M is finite.

Proposition 4.3. *If M is a finite monoid, then*

$$|E(M \wr \mathcal{T}_n)| = \sum_{k=1}^{n} \binom{n}{k} \sum_{\substack{e_1,\ldots,e_k \\ \in E(M)}} \left(\sum_{i=1}^{k} |Me_i| \right)^{n-k}.$$

Proof. To specify an idempotent $(\mathbf{a}, \alpha) \in E(M \wr \mathcal{T}_n)$, we first choose $\mathrm{im}(\alpha)$, of size k (say), in $\binom{n}{k}$ ways. Write $\mathrm{im}(\alpha) = \{j_1, \ldots, j_k\}$, where $j_1 < \cdots < j_k$. We then choose idempotents $a_{j_1} = e_1, \ldots, a_{j_k} = e_k \in E(M)$; see Remark 4.2. We then choose the preimages $j_1\alpha^{-1}, \ldots, j_k\alpha^{-1}$, of sizes $l_1+1, \ldots, l_k+1$ (say), in $\binom{n-k}{l_1,\ldots,l_k}$ ways, noting that $j_i\alpha = j_i$ for each i. For each $1 \leq i \leq k$, and for each $q \in j_i\alpha^{-1} \setminus \{j_i\}$, we must choose $a_q \in M$ so that $a_q e_i = a_q$: i.e., $a_q \in Me_i$. So there are $|Me_i|^{l_i}$ ways to choose $\{a_q : q \in j_i\alpha^{-1} \setminus \{j_i\}\}$. Multiplying and adding these values, as appropriate, gives

$$|E(M \wr \mathcal{T}_n)| = \sum_{k=1}^{n} \binom{n}{k} \sum_{\substack{e_1,\ldots,e_k \\ \in E(M)}} \sum_{\substack{(l_1,\ldots,l_k) \in \mathbb{N}^k \\ l_1+\cdots+l_k = n-k}} \binom{n-k}{l_1, \ldots, l_k} \prod_{i=1}^{k} |Me_i|^{l_i}.$$

The multinomial formula then completes the proof. □

Remark 4.4. In the case that $M = \mathcal{T}_m$ for some m, Proposition 4.3 specialises to a much simpler version of [17, Proposition 3.1]; the latter gives a formula for $|E(\mathcal{T}_m \wr \mathcal{T}_n)|$ that bears more similarity to the displayed equation in the proof of Proposition 4.3. A formula for $|E(M \wr \mathrm{Sing}_n)|$ may be obtained from $|E(M \wr \mathcal{T}_n)|$ by subtracting $|E(M)|^n$ (i.e., the $k = n$ term in the sum from Proposition 4.3).

Proposition 4.3 simplifies substantially in the case that the finite monoid M has only a single idempotent; this occurs precisely when M is a finite group.

Corollary 4.5. *If G is a finite group, then* $|E(G \wr \mathcal{T}_n)| = \displaystyle\sum_{k=1}^{n} \binom{n}{k} k^{n-k} |G|^{n-k}$. □

Remark 4.6. In the case that $|G| = 1$, $G \wr \mathcal{T}_n \cong \mathcal{T}_n$, and Corollary 4.5 reduces to the formula in Proposition 2.1.

4.2 Generation and idempotent generation in $M \wr \mathrm{Sing}_n$

The statement and proof of the next result refer to the sets $\mathcal{X} \subseteq \mathcal{X}_1 \subseteq \mathcal{X}_2 \subseteq \mathcal{X}_n \subseteq M \wr \mathrm{Sing}_n$ defined at the beginning of Section 4, and to the preorder $\leq_{\mathscr{L}}$ and relation $\mathscr{L}$ defined (on M) as in Section 2. Recall that $M/\mathscr{L}$ denotes the partially ordered set of all $\mathscr{L}$-classes of M. We denote by δ the diagonal equivalence $\{(i,i) : i \in \mathbf{n}\}$. Parts (ii) and (iii) of the next result are contained in [9, Theorems 5.5 and 5.7], but they are crucial in what follows, and we provide short proofs for convenience. We note that the corresponding statements from [9] are dual to ours, because we compose maps from left to right. Note also that

$$\begin{aligned} M/\mathscr{L} \text{ is a chain } &\Leftrightarrow \text{ the principal left ideals of } M \text{ form a chain under inclusion} \\ &\Leftrightarrow \text{ all finitely generated left ideals of } M \text{ are principal.} \end{aligned}$$

Theorem 4.7. *If M is any monoid, then*

(i) $M \wr \mathrm{Sing}_n = \langle \mathcal{X}_n \rangle = \langle \mathcal{X}_2 \rangle$;

(ii) $\langle E(M \wr \mathrm{Sing}_n) \rangle = \langle \mathcal{X}_1 \rangle = \{(\mathbf{a}, \alpha) \in M \wr \mathrm{Sing}_n : a_i \leq_{\mathscr{L}} a_j$ *for some* $(i,j) \in \ker(\alpha) \setminus \delta\}$; *and*

(iii) $M \wr \mathrm{Sing}_n$ *is idempotent generated if and only if $M/\mathscr{L}$ is a chain.*

Proof. Theorems 2.2 and 3.1 quickly give $M \wr \mathrm{Sing}_n = \langle \mathscr{X}_n \rangle$. To complete the proof of (i), since $\mathscr{X}_2 \subseteq \mathscr{X}_n$, it suffices to prove that $\mathscr{X}_n \subseteq \langle \mathscr{X}_2 \rangle$, so let $\varepsilon_{ij;\mathbf{a}} \in \mathscr{X}_n$. Relabelling the elements of $\mathbf{n}$, if necessary, we may assume that $(i,j) = (1,2)$. We show in Figure 3 that $\varepsilon_{12;\mathbf{a}} = \varepsilon_{12;a_1,a_2}\varepsilon_{32;a_3,1}\cdots\varepsilon_{n2;a_n,1}$.

For (ii), put

$$\Sigma = \{(\mathbf{a},\alpha) \in M \wr \mathrm{Sing}_n : a_i \leq_{\mathscr{L}} a_j \text{ for some } (i,j) \in \ker(\alpha) \setminus \delta\}.$$

Since $\mathscr{X}_1 \subseteq E(M \wr \mathrm{Sing}_n)$, it suffices to show that: (a) $\langle E(M \wr \mathrm{Sing}_n)\rangle \subseteq \Sigma$; and (b) $\Sigma \subseteq \langle \mathscr{X}_1 \rangle$.

For (a), let $(\mathbf{a},\alpha) \in \langle E(M \wr \mathrm{Sing}_n)\rangle$. So $(\mathbf{a},\alpha) = (\mathbf{b},\beta)(\mathbf{a},\alpha) = (\mathbf{b}(\beta\cdot\mathbf{a}),\beta\alpha)$ for some $(\mathbf{b},\beta) \in E(M \wr \mathrm{Sing}_n)$. Since $\beta \in \mathrm{Sing}_n$, we may choose some $i \in \mathbf{n} \setminus \mathrm{im}(\beta)$. Put $j = i\beta$, noting also that $j = j\beta$ (since β is an idempotent). Examining the ith coordinate of $\mathbf{a} = \mathbf{b}(\beta\cdot\mathbf{a})$ gives $a_i = b_i a_{i\beta} = b_i a_j \leq_{\mathscr{L}} a_j$. Since $(i,j) \in \ker(\beta) \subseteq \ker(\beta\alpha) = \ker(\alpha)$, this completes the proof of (a).

For (b), let $(\mathbf{a},\alpha) \in \Sigma$. Relabelling the elements of $\mathbf{n}$, if necessary, we may assume that $(1,2) \in \ker(\alpha)$ and $a_1 \leq_{\mathscr{L}} a_2$, so that $a_1 = xa_2$ for some $x \in M$. Note that $(\mathbf{a},\alpha) = \varepsilon_{12;\mathbf{a}}\alpha$ (thinking of $\alpha \in \mathrm{Sing}_n$ as an element of $M \wr \mathrm{Sing}_n$). Since $\mathrm{Sing}_n = \langle \mathscr{X} \rangle \subseteq \langle \mathscr{X}_1 \rangle$, it remains to show that $\varepsilon_{12;\mathbf{a}} \in \langle \mathscr{X}_1 \rangle$. As in Part (i), we have $\varepsilon_{12;\mathbf{a}} = \varepsilon_{12;a_1,a_2}\varepsilon_{32;a_3,1}\cdots\varepsilon_{n2;a_n,1}$, and the proof of (b) is complete, after noting that

$$\varepsilon_{12;a_1,a_2} = \varepsilon_{21;x}\varepsilon_{12;a_2} \qquad \text{and} \qquad \varepsilon_{k2;a_k,1} = \varepsilon_{2k;a_k}\varepsilon_{k2;1} \qquad \text{for all } 3 \leq k \leq n.$$

For (iii), suppose first that $M/\mathscr{L}$ is a chain. Consider an arbitrary element $(\mathbf{a},\alpha) \in M \wr \mathrm{Sing}_n$, and choose any $(i,j) \in \ker(\alpha) \setminus \delta$. Then either $a_i \leq_{\mathscr{L}} a_j$ or $a_j \leq_{\mathscr{L}} a_i$ (since $M/\mathscr{L}$ is a chain). Part (ii) then gives $(\mathbf{a},\alpha) \in \langle E(M \wr \mathrm{Sing}_n)\rangle$, showing that $M \wr \mathrm{Sing}_n$ is idempotent generated. Conversely, suppose $M \wr \mathrm{Sing}_n$ is idempotent generated, and let $a,b \in M$ be arbitrary. Then $\varepsilon_{12;ab} \in \Sigma$, by (ii). As we clearly have $\ker(\varepsilon_{12}) \setminus \delta = \{(1,2),(2,1)\}$, it follows that $a \leq_{\mathscr{L}} b$ or $b \leq_{\mathscr{L}} a$. This completes the proof of (iii). $\square$

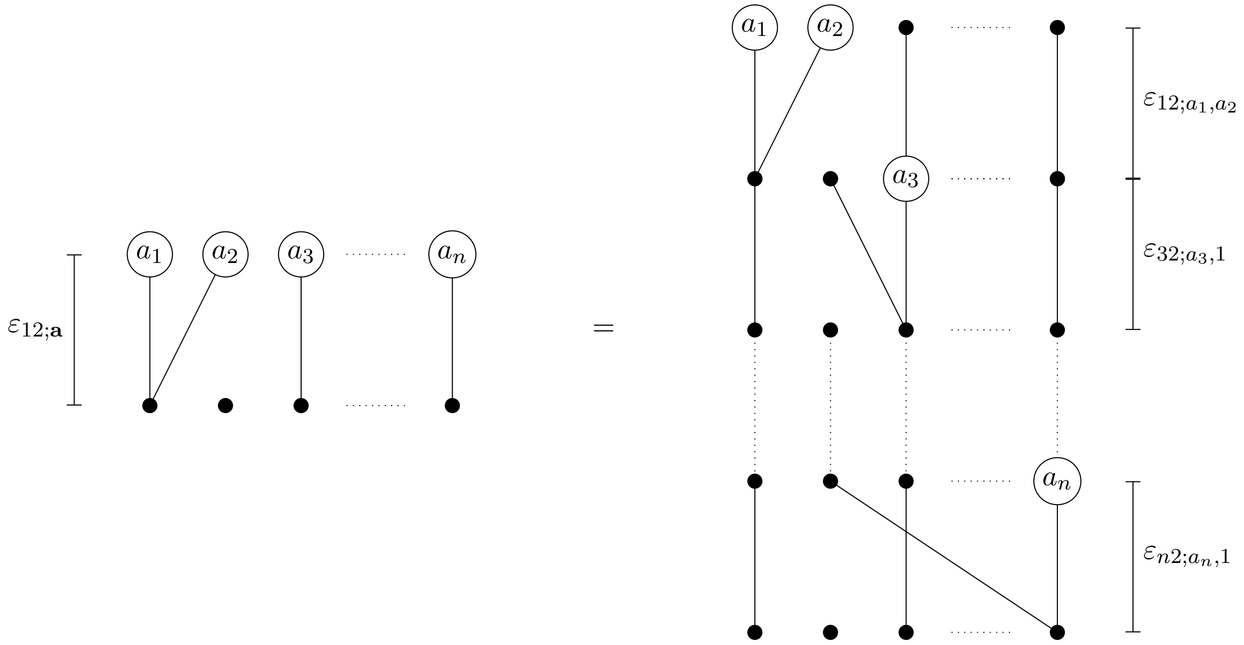


Figure 3: Diagrammatic proof that that $\varepsilon_{12;\mathbf{a}} = \varepsilon_{12;a_1,a_2}\varepsilon_{32;a_3,1}\cdots\varepsilon_{n2;a_n,1}$.

Remark 4.8. Recall from Theorem 2.2 that Sing_n is generated by its idempotents of rank $n-1$. Parts (ii) and (iii) of Theorem 4.7 show that when $M \wr \mathrm{Sing}_n$ is idempotent generated (i.e., when $M/\mathscr{L}$ is a chain), it is generated by idempotents whose underlying (idempotent) transformation has rank $n-1$. See also [36, Theorem 4.2] and [37, Theorem 2.1].

The elements of $\langle E(M \wr \mathcal{T}_n)\rangle$ were characterised in [9, Theorem 5.6]; the description is a little more technical than Theorem 4.7(ii). The next result (which was not proved in [9]) characterises the monoids M for which $\langle E(M \wr \mathcal{T}_n)\rangle$ takes on a particularly simple form. For its statement, recall that we write 1 for the identity element of any monoid (including $M \wr \mathcal{T}_n$).

Corollary 4.9. *For any monoid M, the following are equivalent:*

(i) $\langle E(M \wr \mathcal{T}_n)\rangle = \{1\} \cup (M \wr \mathrm{Sing}_n)$; *and*

(ii) $M/\mathscr{L}$ *is a chain and* $E(M) = \{1\}$.

Proof. Suppose first that (i) holds. If there was a non-identity idempotent $a \in E(M) \setminus \{1\}$, then $((a, 1, \ldots, 1), 1) \in E(M \wr \mathcal{T}_n) \subseteq \{1\} \cup (M \wr \mathrm{Sing}_n)$, a contradiction. It follows that $E(M) = \{1\}$. Now suppose $a, b \in M$ are arbitrary. By assumption, we may write $\varepsilon_{12;ab} = (\mathbf{a}_1, \alpha_1) \cdots (\mathbf{a}_k, \alpha_k)$ for some idempotents $(\mathbf{a}_1, \alpha_1), \ldots, (\mathbf{a}_k, \alpha_k) \in E(M \wr \mathcal{T}_n)$. If any $\alpha_i = 1$, then Lemma 4.1 and $E(M) = \{1\}$ gives $\mathbf{a}_i = (1, \ldots, 1)$; in this case, the factor $(\mathbf{a}_i, \alpha_i)$ is the identity of $M \wr \mathcal{T}_n$, and so may be cancelled from the product. After cancelling all such trivial factors, we deduce that $\varepsilon_{12;ab} \in \langle E(M \wr \mathrm{Sing}_n)\rangle$. Since $\ker(\varepsilon_{12}) \setminus \delta = \{(1, 2), (2, 1)\}$, Theorem 4.7(ii) gives $a \leq_{\mathscr{L}} b$ or $b \leq_{\mathscr{L}} a$. So $M/\mathscr{L}$ is a chain.

Conversely, suppose (ii) holds. First note that $\{1\} \cup (M \wr \mathrm{Sing}_n) = \{1\} \cup \langle E(M \wr \mathrm{Sing}_n)\rangle \subseteq \langle E(M \wr \mathcal{T}_n)\rangle$, by Theorem 4.7(iii). Conversely, suppose $(\mathbf{a}_1, \alpha_1), \ldots, (\mathbf{a}_k, \alpha_k) \in E(M \wr \mathcal{T}_n)$, and put $(\mathbf{a}, \alpha) = (\mathbf{a}_1, \alpha_1) \cdots (\mathbf{a}_k, \alpha_k)$. Again, $E(M) = \{1\}$ tells us that any factor with $\alpha_i = 1$ is the identity of $M \wr \mathcal{T}_n$. If $\alpha_i = 1$ for all i, then $(\mathbf{a}, \alpha) = 1$; otherwise, we cancel all such trivial factors and deduce that $(\mathbf{a}, \alpha) \in \langle E(M \wr \mathrm{Sing}_n)\rangle$. □

4.3 Rank and idempotent rank in $M \wr \mathrm{Sing}_n$

Recall that the rank (resp., idempotent rank) of a semigroup (resp., idempotent generated semigroup), denoted $\mathrm{rank}(S)$ (resp., $\mathrm{idrank}(S)$), is the minimum size of a generating set (resp., idempotent generating set) for S. In this section, we prove a number of results concerning the rank and idempotent rank of $M \wr \mathrm{Sing}_n$.

For $1 \leq i < j \leq n$, write η_{ij} for the equivalence relation on $\mathbf{n}$ with unique non-trivial equivalence class $\{i, j\}$. In particular, $\eta_{ij} = \ker(\varepsilon_{ij}) = \ker(\varepsilon_{ji})$. For several proofs in this section, recall that a directed graph Γ with no loops is a *tournament* if, for every pair of distinct vertices x, y, Γ has exactly one of the edges (x, y) or (y, x). Note that a tournament on vertex set $\mathbf{n}$ has $\binom{n}{2}$ edges if $n \geq 3$ (and that there are no strongly connected tournaments on 2 vertices).

Proposition 4.10. *If M is a finite monoid with group of units G, then*

$$(2|M| - |G|) \times \binom{n}{2} \leq \mathrm{rank}(M \wr \mathrm{Sing}_n) \leq \begin{cases} |M|^2 + 1 & \text{if } n = 2 \\ |M|^2 \times \binom{n}{2} & \text{if } n \geq 3. \end{cases}$$

Proof. Put

$$\Lambda = \begin{cases} \{\varepsilon_{12;ab} : a, b \in M\} \cup \{\varepsilon_{21;11}\} & \text{if } n = 2 \\ \{\varepsilon_{ij;ab} : (i, j) \in E_\Gamma,\ a, b \in M\} & \text{if } n \geq 3, \end{cases}$$

where Γ is a strongly connected tournament with vertex set $\mathbf{n}$ and edge set E_Γ in the $n \geq 3$ case. To establish the stated upper bound, it suffices to show that $M \wr \mathrm{Sing}_n = \langle\Lambda\rangle$; by Theorem 4.7(i), it is enough to show that $\mathcal{X}_2 \subseteq \langle\Lambda\rangle$. With this in mind, let $i, j \in \mathbf{n}$ with $i \neq j$, and let $a, b \in M$. We must show that $\varepsilon_{ij;ab}, \varepsilon_{ji;ab} \in \langle\Lambda\rangle$. Without loss of generality, we may assume that $\varepsilon_{ij;ab} \in \Lambda$ (so $(i, j) = (1, 2)$ if $n = 2$, or $(i, j) \in E_\Gamma$ if $n \geq 3$). Note also that $\varepsilon_{ij;ba} \in \Lambda$. Next, note that $\Lambda \cap \mathrm{Sing}_n$ is a generating set for Sing_n (regarded as a subsemigroup of $M \wr \mathrm{Sing}_n$, as usual), by Theorem 2.3(ii). In particular, $\varepsilon_{ji;11} \in \langle\Lambda\rangle$. But then $\varepsilon_{ji;ab} = \varepsilon_{ij;ba}\varepsilon_{ji;11} \in \langle\Lambda\rangle$, as required.

To establish the stated lower bound, suppose $M \wr \mathrm{Sing}_n = \langle\Lambda\rangle$. Let $1 \leq i < j \leq n$, let $a \in M$, and consider an expression $\varepsilon_{ij;a} = (\mathbf{a}, \alpha)(\mathbf{a}_1, \alpha_1) \cdots (\mathbf{a}_k, \alpha_k)$, where $k \geq 0$ and $(\mathbf{a}, \alpha), (\mathbf{a}_1, \alpha_1), \ldots, (\mathbf{a}_k, \alpha_k) \in \Lambda$. Since $\varepsilon_{ij} = \alpha\alpha_1 \cdots \alpha_k$, we see that $\ker(\alpha) = \eta_{ij}$. Write $(\mathbf{b}, \beta) = (\mathbf{a}_1, \alpha_1) \cdots (\mathbf{a}_k, \alpha_k)$, so that $\varepsilon_{ij;a} = (\mathbf{a}(\alpha \cdot \mathbf{b}), \alpha\beta)$. In particular, examining the ith and jth coordinates of $\mathbf{a}(\alpha \cdot \mathbf{b})$, we see that $1 = a_i b_{i\alpha}$ and $a = a_j b_{j\alpha} = a_j b_{i\alpha}$. It follows that $a_i \in G$ (as M is finite) and $a_i^{-1} = b_{i\alpha}$, so that $a = a_j a_i^{-1}$. To summarise, Λ contains an element $(\mathbf{a}, \alpha)$ with $\ker(\alpha) = \eta_{ij}$, $a_i \in G$, and $a_j a_i^{-1} = a$. A similar argument (considering a factorisation of $\varepsilon_{ji;a}$) shows that Λ contains an element $(\mathbf{c}, \gamma)$ with $\ker(\gamma) = \eta_{ij}$, $c_j \in G$, and $c_i c_j^{-1} = a$. If $a \in M \setminus G$, then

$(\mathbf{a},\alpha) \neq (\mathbf{c},\gamma)$, or else then $a = a_j a_i^{-1} = c_j a_i^{-1} \in G$ (though might possibly be the case that $(\mathbf{a},\alpha) = (\mathbf{c},\gamma)$ if $a \in G$). In particular, Λ contains at least $2|M \setminus G| + |G| = 2|M| - |G|$ elements whose underlying transformation has kernel η_{ij}. Since this is true for all $1 \leq i < j \leq n$, it follows that $|\Lambda| \geq (2|M|-|G|) \times \binom{n}{2}$. Since this is true for any generating set, the proof is complete. □

Although the next result is a special case of the one immediately following it, it will be convenient to state and prove it separately.

Proposition 4.11. *If G is a finite group, then*

$$\operatorname{rank}(G \wr \operatorname{Sing}_n) = \operatorname{idrank}(G \wr \operatorname{Sing}_n) = \begin{cases} 2 & \text{if } n = 2 \text{ and } |G| = 1 \\ |G| \times \binom{n}{2} & \text{otherwise.} \end{cases}$$

Proof. If $n = 2$ and $|G| = 1$, then $G \wr \operatorname{Sing}_n \cong \operatorname{Sing}_2$ is a right zero semigroup of size 2, so the result is clear. For the rest of the proof, suppose at least one of $n \geq 3$ or $|G| \geq 2$ holds. By Proposition 4.10, the proof will be complete if we can produce an idempotent generating set for $G \wr \operatorname{Sing}_n$ of the specified size. With this in mind, put

$$\Lambda = \begin{cases} \{\varepsilon_{12;a} : a \in G \setminus \{1\}\} \cup \{\varepsilon_{21;1}\} & \text{if } n = 2 \\ \{\varepsilon_{ij;a} : (i,j) \in E_\Gamma,\ a \in G\} & \text{if } n \geq 3, \end{cases}$$

where Γ is a strongly connected tournament with vertex set $\mathbf{n}$ and edge set E_Γ in the $n \geq 3$ case. Since Λ has the required size, it suffices to show that $\mathscr{X}_1 \subseteq \langle \Lambda \rangle$, by Theorem 4.7(ii) and (iii). As in the proof of Proposition 4.10, we immediately see that $\langle \Lambda \rangle$ contains $\mathscr{X}$ if $n \geq 3$. We claim that this is also the case for $n = 2$. Indeed, if $n = 2$, then we already have $\varepsilon_{21;1} \in \Lambda$, while for any $a \in G \setminus \{1\}$ with $a^k = 1$ (and $k \geq 1$), we have $\varepsilon_{12;1} = (\varepsilon_{12;a,a})^k = (\varepsilon_{21;1}\varepsilon_{12;a})^k \in \langle \Lambda \rangle$. So far we have shown that $\mathscr{X} \subseteq \langle \Lambda \rangle$. Let $i, j \in \mathbf{n}$ with $i \neq j$, and let $a \in G \setminus \{1\}$. The proof will be complete if we can show that $\varepsilon_{ij;a}, \varepsilon_{ji;a} \in \langle \Lambda \rangle$. Without loss of generality, we may assume that $\varepsilon_{ij;a} \in \Lambda$, which also implies that $\varepsilon_{ij;a^{-1}} \in \Lambda$ (by definition of Λ). But then $\varepsilon_{ji;a} = \varepsilon_{ij;a^{-1}}\varepsilon_{ji;1}\varepsilon_{ij;a}\varepsilon_{ji;1} \in \langle \Lambda \rangle$. □

The next result extends Proposition 4.11 to the case of an arbitrary finite idempotent generated singular wreath product. In parts of its proof, we use some general results of Gray [44] on (idempotent) ranks of completely 0-simple semigroups; see also [30, 47]. We also make use of Green's relations, as defined in Section 2.

Theorem 4.12. *If M is a finite monoid with group of units G, and if $M/\mathscr{L}$ is a chain, then*

$$\operatorname{rank}(M \wr \operatorname{Sing}_n) = \begin{cases} 2 & \text{if } n = 2 \text{ and } |M| = 1 \\ (2|M| - |G|)\binom{n}{2} & \text{otherwise,} \end{cases}$$

and

$$\operatorname{idrank}(M \wr \operatorname{Sing}_n) = \begin{cases} 2|M| & \text{if } n = 2 \text{ and } |G| = 1 \\ (2|M| - |G|)\binom{n}{2} & \text{otherwise.} \end{cases}$$

Proof. The case in which $M = G$ is covered in Proposition 4.11, so suppose $M \neq G$. In particular, $|M| \geq 2$. We break the proof up into three cases.

Case 1. Suppose first that $n \geq 3$. The proof will be complete (in this case) if we can produce an idempotent generating set for $M \wr \operatorname{Sing}_n$ of size $(2|M| - |G|)\binom{n}{2}$. With this in mind, let Γ be a strongly connected tournament on vertex set $\mathbf{n}$, and put $\Lambda = \Lambda_1 \cup \Lambda_2$, where

$$\Lambda_1 = \{\varepsilon_{ij;a} : i,j \in \mathbf{n},\ i \neq j,\ a \in M \setminus G\} \qquad \text{and} \qquad \Lambda_2 = \{\varepsilon_{ij;a} : (i,j) \in E_\Gamma,\ a \in G\}.$$

The proof of Proposition 4.11 gives $\langle \Lambda_2 \rangle = G \wr \operatorname{Sing}_n$. Consequently, $\langle \Lambda \rangle$ contains $\mathscr{X}_1$, and we are done, by Theorem 4.7.

Case 2. Next suppose that $n = 2$ and $|G| \geq 2$. This time, put $\Lambda = \Lambda_1 \cup \Lambda_2$, where

$$\Lambda_1 = \{\varepsilon_{12;a}, \varepsilon_{21;a} : a \in M \setminus G\} \qquad \text{and} \qquad \Lambda_2 = \{\varepsilon_{12;a} : a \in G \setminus \{1\}\} \cup \{\varepsilon_{21;1}\}.$$

Again, the proof of Proposition 4.11 gives $\langle \Lambda_2 \rangle = G \wr \mathrm{Sing}_2$, and it quickly follows that Λ is an idempotent generating set of the required size, completing the proof in this case.

Case 3. Finally, suppose $n = 2$ and $|G| = 1$, noting that $M \wr \mathrm{Sing}_2 = \mathcal{X}_2 = \{\varepsilon_{12;ab}, \varepsilon_{21;ab} : a, b \in M\}$. It is easy to check that for any $a \in M$, and for $(i, j) = (1, 2)$ or $(2, 1)$,

$$R_{\varepsilon_{ij;1a}} = R_{\varepsilon_{ji;a1}} = \{\varepsilon_{ij;1a}, \varepsilon_{ji;a1}\} \qquad \text{and} \qquad L_{\varepsilon_{ij;1a}} = L_{\varepsilon_{ij;a1}} = \{\varepsilon_{ij;1b}, \varepsilon_{ij;b1} : b \in M\}.$$

It follows that the set

$$J = \{\varepsilon_{12;ab}, \varepsilon_{21;ab} : a, b \in M,\ 1 \in \{a, b\}\}$$

is a $\mathscr{D} = \mathscr{J}$-class of $M \wr \mathrm{Sing}_2$. It is clear that $\mathcal{X}_1 = E(J)$, so it follows that $M \wr \mathrm{Sing}_2 = \langle J \rangle = \langle E(J) \rangle$. It then follows from [30, Lemma 3.2] that

$$\mathrm{rank}(M \wr \mathrm{Sing}_2) = \mathrm{rank}(J^\circ) \qquad \text{and} \qquad \mathrm{idrank}(M \wr \mathrm{Sing}_2) = \mathrm{idrank}(J^\circ),$$

where J° is the *principal factor* of J: i.e., the semigroup with underlying set $J \cup \{0\}$ and operation $\circ$ defined, for $x, y \in J \cup \{0\}$, by

$$x \circ y = \begin{cases} xy & \text{if } x, y, xy \in J \\ 0 & \text{otherwise.} \end{cases}$$

Since $M \wr \mathrm{Sing}_2$ is generated by $\mathcal{X}_1 = E(J)$, it follows that J° is idempotent generated. It then follows from [44, Lemma 2.3] that $\mathrm{rank}(J^\circ)$ is equal to the larger of $|J/\mathscr{R}|$ and $|J/\mathscr{L}|$. Write $M = \{a_1, \ldots, a_k\}$, where $k = |M|$ and $a_1 = 1$. From the above calculations, we see that the $\mathscr{L}$- and $\mathscr{R}$-classes contained in J are

$$L_1 = L_{\varepsilon_{12;11}}, \quad L_2 = L_{\varepsilon_{21;11}} \qquad \text{and} \qquad R_{1s} = R_{\varepsilon_{12;1a_s}}, \quad R_{s1} = R_{\varepsilon_{12;a_s1}} \qquad \text{for each } 1 \leq s \leq k,$$

so that $|J/\mathscr{R}| = 2k - 1$ and $|J/\mathscr{L}| = 2$, giving $\mathrm{rank}(J^\circ) = 2k - 1$. To complete the proof, we need to show that $\mathrm{idrank}(J^\circ) = 2k$. To do this, we use some more ideas from [44]. For a subset $F \subseteq \mathcal{X}_1 = E(J)$, define a (bipartite) graph Δ_F as follows: the vertex set of Δ_F is $(J/\mathscr{R}) \cup (J/\mathscr{L})$, and there is an (undirected) edge between $R \in J/\mathscr{R}$ and $L \in J/\mathscr{L}$ if and only if $(R \cap L) \cap F \neq \emptyset$. Such a graph Δ_F is a subgraph of $\Delta_{E(J)}$, which is pictured in Figure 4. By [44, Lemma 2.5], $J^\circ = \langle F \rangle$ if and only if Δ_F is connected. Since the removal of any edge from $\Delta_{E(J)}$ disconnects the graph, it follows that no proper subset F of $E(J)$ generates J°. So $\mathrm{idrank}(J^\circ) = |E(J)| = 2k$, and the proof is complete. □

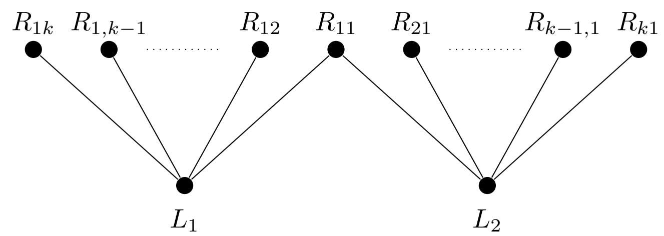


Figure 4: The graph $\Delta_{E(J)}$; see the proof of Theorem 4.12 for more details.

Remark 4.13. Theorem 4.12 shows that when $M \wr \mathrm{Sing}_n$ is idempotent generated, its rank and idempotent rank are equal unless $n = 2$ and $|G| = 1 \neq |M|$, in which case the rank and idempotent rank are equal to $2|M| - 1$ and $2|M|$, respectively. It would be interesting to calculate exact values for $\mathrm{rank}(M \wr \mathrm{Sing}_n)$ for various classes of finite monoids M where $M/\mathscr{L}$ is not a chain.

We now give some information about the (idempotent) rank of $M \wr \mathrm{Sing}_n$ for infinite M. A monoid M has a natural (right) *diagonal* action on the set $M \times M$, defined by $(a, b) \cdot c = (ac, bc)$, for all $a, b, c \in M$. We say this action is *finitely generated* if there exists a finite subset $\Omega \subseteq M \times M$ such that $\Omega \cdot M = M \times M$. It is easy to check that the diagonal action of an infinite group is never finitely generated.

Theorem 4.14. (i) *If M is an infinite monoid, then $M \wr \mathrm{Sing}_n$ is finitely generated if and only if M is finitely generated (as a semigroup) and the diagonal action of M on $M \times M$ is finitely generated.*

(ii) *If $M \wr \mathrm{Sing}_n$ is not finitely generated (e.g., if M is an infinite group), then* $\mathrm{rank}(M \wr \mathrm{Sing}_n) = |M|$*; if $M/\mathscr{L}$ is also a chain, then* $\mathrm{idrank}(M \wr \mathrm{Sing}_n) = |M|$.

Proof. Suppose M is an infinite monoid. For (i), suppose first that $M \wr \mathrm{Sing}_n$ is generated by a finite set $\Lambda = \{(\mathbf{a}_s, \alpha_s) : 1 \leq s \leq k\}$. For $1 \leq s \leq k$, write $\mathbf{a}_s = (a_{s1}, \ldots, a_{sn})$, and put

$$A = \{a_{st} : 1 \leq s \leq k,\ 1 \leq t \leq n\} \qquad \text{and} \qquad \Omega = \{(a_{s1}, a_{s2}) : 1 \leq s \leq k\},$$

noting that both sets are finite. Let $a \in M$ be arbitrary. Consideration of $\varepsilon_{12;a}$ as a product of elements from Λ shows that $a \in \langle A \rangle$, giving $M = \langle A \rangle$. To show that $M \times M$ is finitely generated as an M-act, let $a, b \in M$ be arbitrary, and consider a product

$$\varepsilon_{12;ab} = (\mathbf{a}_s, \alpha_s)(\mathbf{a}_{s_1}, \alpha_{s_1}) \cdots (\mathbf{a}_{s_r}, \alpha_{s_r}) \qquad \text{where } 1 \leq s, s_1, \ldots, s_r \leq k.$$

From $\varepsilon_{12} = \alpha_s \alpha_{s_1} \cdots \alpha_{s_r}$, we deduce $\ker(\alpha_s) = \eta_{12}$. Writing $(\mathbf{b}, \beta) = (\mathbf{a}_{s_1}, \alpha_{s_1}) \cdots (\mathbf{a}_{s_r}, \alpha_{s_r})$, and examining the first and second coordinates of $\varepsilon_{12;ab} = (\mathbf{a}_s, \alpha_s)(\mathbf{b}, \beta) = (\mathbf{a}_s(\alpha_s \cdot \mathbf{b}), \alpha_s \beta)$, we see that

$$(a, b) = (a_{s1} b_{1\alpha_s}, a_{s2} b_{2\alpha_s}) = (a_{s1} b_{1\alpha_s}, a_{s2} b_{1\alpha_s}) = (a_{s1}, a_{s2}) \cdot b_{1\alpha_s},$$

showing that $M \times M = \Omega \cdot M$.

Conversely, suppose $M = \langle A \rangle$ and $M \times M = \Omega \cdot M$ for finite sets $A \subseteq M$ and $\Omega \subseteq M \times M$. Put

$$\Lambda = \{\varepsilon_{ij;ab} : i, j \in \mathbf{n},\ i \neq j,\ (a, b) \in \Omega\} \cup \{\varepsilon_{ij;a1} : i, j \in \mathbf{n},\ i \neq j,\ a \in A\}.$$

Since Λ is finite, the proof of (i) will be complete (by Theorem 4.7(i)) if we can show that $\langle \Lambda \rangle$ contains $\mathcal{X}_2$. So let $a, b \in M$, and let $i, j \in \mathbf{n}$ with $i \neq j$. Now, $(a, b) = (c, d) \cdot f = (cf, df)$ for some $(c, d) \in \Omega$ and $f \in M$, and we may write $f = f_1 \cdots f_r$ for some $f_1, \ldots, f_r \in A$. But then

$$\varepsilon_{ij;ab} = \varepsilon_{ij;cd} \varepsilon_{ij;f1} = \varepsilon_{ij;cd} \varepsilon_{ij;f_1 1} \cdots \varepsilon_{ij;f_r 1} \in \langle \Lambda \rangle,$$

as required. Part (ii) follows from the fact that $\mathrm{rank}(S) = |S|$ for any non-finitely generated semigroup S. □

Remark 4.15. We have not attempted to give the value (or any estimate) for the rank (or idempotent rank, if applicable) of $M \wr \mathrm{Sing}_n$ in the case that $M \wr \mathrm{Sing}_n$ is infinite but finitely generated. Although the generating set Λ constructed in the second paragraph of the proof of Theorem 4.14(i) leads to an upper bound for $\mathrm{rank}(M \wr \mathrm{Sing}_n)$, Λ is certainly not of minimal size, in general. It would be interesting to calculate exact values for the rank (and idempotent rank, if appropriate) of $M \wr \mathrm{Sing}_n$ for various classes of finitely generated infinite monoids M with finitely generated diagonal action.

Remark 4.16. Finite generation of the diagonal action plays an important role in [73]. However, the wreath products studied there are different to those studied here, so Theorem 4.14(i) appears to be independent of the results from [73]. See also [39,40,72], where diagonal actions are studied in detail, and several interesting examples discussed.

5 Presentations for $M \wr \mathrm{Sing}_n$

We now turn to the main topic of the paper: namely, the task of finding presentations for the singular wreath products $M \wr \mathrm{Sing}_n$. In Section 5.1, we give a presentation in terms of the generating set $\mathcal{X}_2$ (see Theorem 5.2). In Section 5.2, we give a presentation in terms of the idempotent generating set $\mathcal{X}_1$ in the case that $M/\mathscr{L}$ is a chain (see Theorem 5.9). Our strategy is to begin with the general presentation for semidirect products (Theorem 3.1) in order to deduce a presentation for $M \wr \mathrm{Sing}_n$ in terms of the (very large) generating set $\mathcal{X}_n$ (Corollary 5.1), and to then reduce this to the above-mentioned simpler presentations, using Tietze transformations. Each of the presentations we give in this section extends the presentation for Sing_n stated in Theorem 2.4.

5.1 A presentation for $M \wr \mathrm{Sing}_n$

Since $M \wr \mathrm{Sing}_n = M^n \rtimes \mathrm{Sing}_n$ is a semidirect product, Theorem 3.1 allows us to write down a presentation for $M \wr \mathrm{Sing}_n$ in terms of the presentation $\langle X : R\rangle$ for Sing_n from Theorem 2.4. In order to state this presentation (in Corollary 5.1 below), define an alphabet

$$X_n = \{e_{ij;\mathbf{a}} : i,j \in \mathbf{n},\ i \neq j,\ \mathbf{a} \in M^n\},$$

an epimorphism

$$\phi_n : X_n^+ \to M \wr \mathrm{Sing}_n : e_{ij;\mathbf{a}} \mapsto \varepsilon_{ij;\mathbf{a}},$$

and let R_n be the set of relations (identifying a letter $e_{ij} \in X$ with $e_{ij;(1,\ldots,1)} \in X_n$, as in Section 3)

$$\begin{aligned}
e_{ij;\mathbf{a}}e_{ij} = e_{ij;\mathbf{a}} &= e_{ji;\mathbf{a}}e_{ij} && \text{for } \mathbf{a} \in M^n \text{ and distinct } i,j && (\mathrm{R1})_n\\
e_{ij;\mathbf{a}}e_{kl} &= e_{kl;\mathbf{a}}e_{ij} && \text{for } \mathbf{a} \in M^n \text{ and distinct } i,j,k,l && (\mathrm{R2})_n\\
e_{ik;\mathbf{a}}e_{jk} &= e_{ik;\mathbf{a}} && \text{for } \mathbf{a} \in M^n \text{ and distinct } i,j,k && (\mathrm{R3})_n\\
e_{ij;\mathbf{a}}e_{ik} = e_{ik;\mathbf{a}}e_{ij} &= e_{jk;\mathbf{a}}e_{ij} && \text{for } \mathbf{a} \in M^n \text{ and distinct } i,j,k && (\mathrm{R4})_n\\
e_{ki;\mathbf{a}}e_{ij}e_{jk} &= e_{ik;\mathbf{a}}e_{kj}e_{ji}e_{ik} && \text{for } \mathbf{a} \in M^n \text{ and distinct } i,j,k && (\mathrm{R5})_n\\
e_{ki;\mathbf{a}}e_{ij}e_{jk}e_{kl} &= e_{ik;\mathbf{a}}e_{kl}e_{li}e_{ij}e_{jl} && \text{for } \mathbf{a} \in M^n \text{ and distinct } i,j,k,l && (\mathrm{R6})_n\\
e_{ij;\mathbf{a}}e_{kl;\mathbf{b}} &= e_{ij;\mathbf{c}}e_{kl} && \text{for } \mathbf{a},\mathbf{b} \in M^n \text{ and any } i,j,k,l, && (\mathrm{R7})_n
\end{aligned}$$

where in $(\mathrm{R7})_n$, $\mathbf{c} = \mathbf{a}(\varepsilon_{ij} \cdot \mathbf{b}) = (c_1,\ldots,c_n)$ satisfies $c_j = a_jb_i$ and $c_k = a_kb_k$ for $k \neq j$. Note that i,j,k,l are not assumed to be distinct (apart from $i \neq j$ and $k \neq l$) in $(\mathrm{R7})_n$. As noted above, the next result is a special case of Theorem 3.1.

Corollary 5.1. *The semigroup $M \wr \mathrm{Sing}_n$ has presentation $\langle X_n : R_n\rangle$ via ϕ_n.* □

The presentation $\langle X_n : R_n\rangle$ utilises the large generating set $\mathcal{X}_n$. Consequently, our next goal is to simplify this presentation to obtain a presentation utilising the smaller generating set $\mathcal{X}_2 \subseteq \mathcal{X}_n$. With this in mind, define an alphabet

$$X_2 = \{e_{ij;ab} : i,j \in \mathbf{n},\ i \neq j,\ a,b \in M\},$$

an epimorphism

$$\phi_2 : X_2^+ \to M \wr \mathrm{Sing}_n : e_{ij;ab} \mapsto \varepsilon_{ij;ab},$$

and let R_2 be the set of relations

$$\begin{aligned}
e_{ij;ab}e_{ij;cd} = e_{ij;ac,bc} &= e_{ji;ba}e_{ij;dc} && \text{for } a,b,c,d \in M \text{ and distinct } i,j && (\mathrm{R1})_2\\
e_{ij;ab}e_{kl;cd} &= e_{kl;cd}e_{ij;ab} && \text{for } a,b,c,d \in M \text{ and distinct } i,j,k,l && (\mathrm{R2})_2\\
e_{ik;ab}e_{jk;1c} &= e_{ik;ab} && \text{for } a,b,c \in M \text{ and distinct } i,j,k && (\mathrm{R3a})_2\\
e_{ik;ab}e_{jk;c1} &= e_{ki;ba}e_{ji;c1}e_{ik;11} && \text{for } a,b,c \in M \text{ and distinct } i,j,k && (\mathrm{R3b})_2\\
e_{ik;aa}e_{jk;b1} &= e_{ik;11}e_{jk;b1}e_{ik;a1} && \text{for } a,b \in M \text{ and distinct } i,j,k && (\mathrm{R3c})_2\\
e_{ij;ab}e_{ik;cd} = e_{ik;ac,d}e_{ij;1,bc} &= e_{jk;bc,d}e_{ij;ac,1} && \text{for } a,b,c,d \in M \text{ and distinct } i,j,k && (\mathrm{R4a})_2\\
e_{ij;c,ad}e_{ik;1,bd} = e_{ik;c,bd}e_{ij;1,ad} &= e_{jk;ab}e_{ij;cd} && \text{for } a,b,c,d \in M \text{ and distinct } i,j,k && (\mathrm{R4b})_2\\
e_{ki}e_{ij}e_{jk} &= e_{ik}e_{kj}e_{ji}e_{ik} && \text{for distinct } i,j,k && (\mathrm{R5})_2\\
e_{ki}e_{ij}e_{jk}e_{kl} &= e_{ik}e_{kl}e_{li}e_{ij}e_{jl} && \text{for distinct } i,j,k,l. && (\mathrm{R6})_2
\end{aligned}$$

Note that the labelling of relations is chosen to reflect the labels of the relations from R (as stated in Section 2), some of which have been split up into separate relations in R_2. Note also that in relations $(\mathrm{R5})_2$ and $(\mathrm{R6})_2$, we identify X as a subset of X_2 via $e_{ij} \equiv e_{ij;11}$; in this way, relations $(\mathrm{R5})_2$ and $(\mathrm{R6})_2$ are really just (R5) and (R6). Finally, note also that in several relations, we have separated the two monoid subscripts on a letter from X_2 with a comma when at least one of the subscripts involves a product; for example, the two monoid subscripts on "$e_{ij;ac,bc}$" in relation $(\mathrm{R1})_2$ are the products ac and bc from M. Our goal in this section is to prove the following.

Theorem 5.2. *The semigroup $M \wr \mathrm{Sing}_n$ has presentation $\langle X_2 : R_2 \rangle$ via ϕ_2.*

Remark 5.3. The "$e_{ik;ac,d}e_{ij;1,bc} = e_{jk;bc,d}e_{ij;ac,1}$" part of $(\mathrm{R4a})_2$ follows from the "$e_{ik;c,bd}e_{ij;1,ad} = e_{jk;ab}e_{ij;cd}$" part of $(\mathrm{R4b})_2$, upon making the substitution $(a, b, c, d) \to (bc, d, ac, 1)$. Similarly, the "$e_{ij;c,ad}e_{ik;1,bd} = e_{ik;c,bd}e_{ij;1,ad}$" part of $(\mathrm{R4b})_2$ follows from the "$e_{ij;ab}e_{ik;cd} = e_{ik;ac,d}e_{ij;1,bc}$" part of $(\mathrm{R4a})_2$. As such, we could replace $(\mathrm{R4a})_2$ and $(\mathrm{R4b})_2$ by just

$$e_{ij;ab}e_{ik;cd} = e_{ik;ac,d}e_{ij;1,bc} \qquad \text{and} \qquad e_{ik;c,bd}e_{ij;1,ad} = e_{jk;ab}e_{ij;cd}.$$

However, in the calculations that follow, it is convenient to leave both relations as they are.

As noted earlier, we will prove Theorem 5.2 by performing a sequence of *Tietze transformations*, beginning with the presentation $\langle X_n : R_n \rangle$ from Corollary 5.1. We will write $\sim_n = R_n^\sharp = \ker(\phi_n)$ for the congruence on X_n^+ generated by R_n, and we think of X_2 as a subset of X_n by identifying $e_{ij;ab} \in X_2$ with $e_{ij;\mathbf{a}} \in X_n$, where $\mathbf{a} = (a_1, \ldots, a_n)$ satisfies $a_i = a$, $a_j = b$ and $a_k = 1$ if $k \notin \{i, j\}$.

One may check (diagrammatically) that $u\phi_n = v\phi_n$ (in $M \wr \mathrm{Sing}_n$) for each relation (u, v) from R_2. We do this for relations $(\mathrm{R1})_2$ and $(\mathrm{R3b})_2$ in Figure 5, and leave the reader to check the rest. In particular, we may add the relations R_2 to the presentation $\langle X_n : R_n \rangle$ to obtain $\langle X_n : R_2 \cup R_n \rangle$.

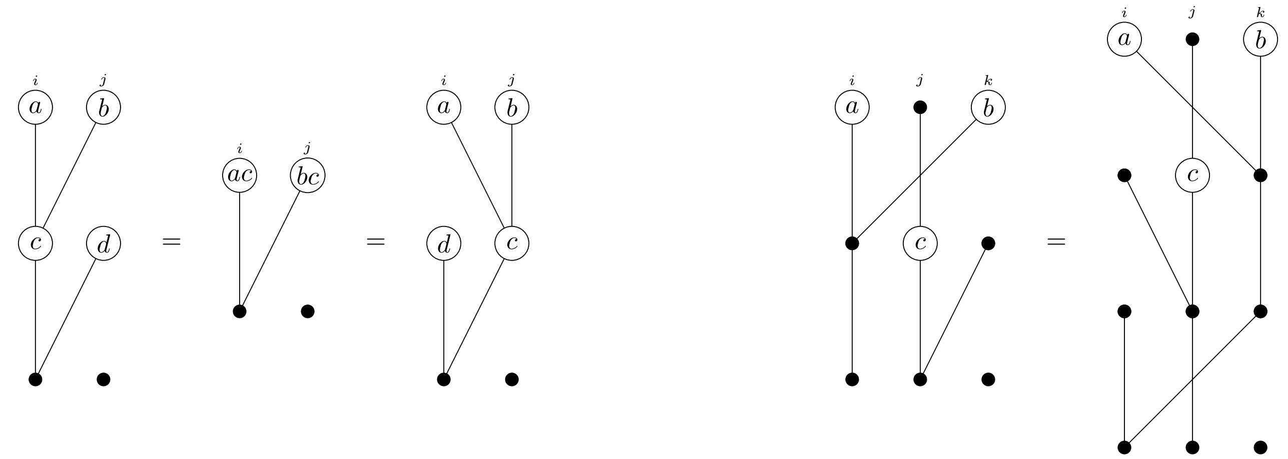


Figure 5: Diagrammatic verification of relations $(\mathrm{R1})_2$, left, and $(\mathrm{R3b})_2$, right. Only the relevant parts of the diagrams have been included, and no ordering on i, j (or i, j, k) is to be implied.

Out next goal is to show that the generators from $X_n \setminus X_2$ may be removed from the presentation. To do this, we define words $E_{ij;\mathbf{a}} \in X_2^+$, for each $i, j \in \mathbf{n}$ with $i \neq j$ and each $\mathbf{a} = (a_1, \ldots, a_n) \in M^n$, as follows. If it happens that $a_k = 1$ for all $k \in \mathbf{n} \setminus \{i, j\}$, then we simply define $E_{ij;\mathbf{a}} = e_{ij;a_ia_j}$. Otherwise, we first write $\mathbf{n} \setminus \{i, j\} = \{i_1, \ldots, i_{n-2}\}$, where $i_1 < \cdots < i_{n-2}$, and define

$$E_{ij;\mathbf{a}} = e_{ij;a_ia_j}e_{i_1j;a_{i_1}1} \cdots e_{i_{n-2}j;a_{i_{n-2}}1}.$$

As shown in Figure 3, we have $E_{ij;\mathbf{a}}\phi_n = \varepsilon_{ij;\mathbf{a}} = e_{ij;\mathbf{a}}\phi_n$. In particular, because $\sim_n = \ker(\phi_n)$, it follows that $e_{ij;\mathbf{a}} \sim_n E_{ij;\mathbf{a}}$. So we may remove each generator $e_{ij;\mathbf{a}} \in X_n \setminus X_2$ from the presentation $\langle X_n : R_2 \cup R_n \rangle$, replacing every occurrence of such a generator in the relations by the word $E_{ij;\mathbf{a}} \in X_2^+$. Since $R_2 \subseteq X_2^+ \times X_2^+$, the only relations modified in this way are those from R_n. We label the relations modified in this way as

$E_{ij;\mathbf{a}}e_{ij} = E_{ij;\mathbf{a}} = E_{ji;\mathbf{a}}e_{ij}$	for $\mathbf{a} \in M^n$ and distinct i, j	$(\mathrm{R1})'_n$
$E_{ij;\mathbf{a}}e_{kl} = E_{kl;\mathbf{a}}e_{ij}$	for $\mathbf{a} \in M^n$ and distinct i, j, k, l	$(\mathrm{R2})'_n$
$E_{ik;\mathbf{a}}e_{jk} = E_{ik;\mathbf{a}}$	for $\mathbf{a} \in M^n$ and distinct i, j, k	$(\mathrm{R3})'_n$
$E_{ij;\mathbf{a}}e_{ik} = E_{ik;\mathbf{a}}e_{ij} = E_{jk;\mathbf{a}}e_{ij}$	for $\mathbf{a} \in M^n$ and distinct i, j, k	$(\mathrm{R4})'_n$
$E_{ki;\mathbf{a}}e_{ij}e_{jk} = E_{ik;\mathbf{a}}e_{kj}e_{ji}e_{ik}$	for $\mathbf{a} \in M^n$ and distinct i, j, k	$(\mathrm{R5})'_n$
$E_{ki;\mathbf{a}}e_{ij}e_{jk}e_{kl} = E_{ik;\mathbf{a}}e_{kl}e_{li}e_{ij}e_{jl}$	for $\mathbf{a} \in M^n$ and distinct i, j, k, l	$(\mathrm{R6})'_n$
$E_{ij;\mathbf{a}}E_{kl;\mathbf{b}} = E_{ij;\mathbf{c}}e_{kl}$	for $\mathbf{a}, \mathbf{b} \in M^n$ and any i, j, k, l,	$(\mathrm{R7})'_n$

and denote the whole set of relations modified in this way by R_n'. So the presentation has now become $\langle X_2 : R_2 \cup R_n'\rangle$. To complete the proof of Theorem 5.2, it remains to show that the relations R_n' can be removed. That is, we need to show that $u \sim_2 v$ for each relation $(u,v) \in R_n'$, where $\sim_2 = R_2^\sharp$ denotes the congruence on X_2^+ generated by R_2. We begin this task with some technical results that will be of use on a number of occasions.

Lemma 5.4. *For any $i,j,k,l \in \mathbf{n}$ with $i \neq j$ and with j,k,l distinct, and for any $a,b,c,d \in M$, we have $e_{ij;ab}e_{kj;c1}e_{lj;d1} \sim_2 e_{ij;ab}e_{lj;d1}e_{kj;c1}$.*

Proof. We have

$$\begin{aligned} e_{ij;ab}e_{kj;c1}e_{lj;d1} &\sim_2 e_{ij;ab}e_{kj;11}e_{kj;c1}e_{lj;d1} && \text{by (R1)}_2 \text{ if } k=i\text{, or (R3a)}_2 \text{ if } k\neq i \\ &\sim_2 e_{ij;ab}e_{kj;cc}e_{lj;d1} && \text{by (R1)}_2 \\ &\sim_2 e_{ij;ab}e_{kj;11}e_{lj;d1}e_{kj;c1} && \text{by (R3c)}_2 \\ &\sim_2 e_{ij;ab}e_{lj;d1}e_{kj;c1} && \text{by (R1)}_2 \text{ or (R3a)}_2. \end{aligned}$$ □

Corollary 5.5. *For any $i,j,k_1,\ldots,k_t \in \mathbf{n}$ with $i \neq j$ and with $j,k_1,\ldots,k_t$ distinct, for any $a,b,c_1,\ldots,c_t \in M$, and for any permutation $\pi \in \mathcal{S}_t$, we have*

$$e_{ij;ab}e_{k_1j;c_11}\cdots e_{k_tj;c_t1} \sim_2 e_{ij;ab}e_{k_{1\pi}j;c_{1\pi}1}\cdots e_{k_{t\pi}j;c_{t\pi}1}.$$

Proof. Consider the subword $e_{k_sj;c_s1}e_{k_{s+1}j;c_{s+1}1}$ in the product $e_{k_1j;c_11}\cdots e_{k_tj;c_t1}$. The letter immediately before this subword is $e_{lj;uv}$ for some $l \in \mathbf{n}\setminus\{j\}$ and some $u,v \in M$. In particular, Lemma 5.4 gives $e_{lj;uv}e_{k_sj;c_s1}e_{k_{s+1}j;c_{s+1}1} \sim_2 e_{lj;uv}e_{k_{s+1}j;c_{s+1}1}e_{k_sj;c_s1}$. So the result is true if π is a simple transposition $(s,s+1)$. Since π is the product of such simple transpositions, the result follows. □

As a special case of the previous result, we may immediately deduce the following.

Corollary 5.6. *Let $i,j \in \mathbf{n}$ with $i \neq j$, and let $\mathbf{a} \in M^n$. If $j_1,\ldots,j_{n-2}$ is any ordering of $\mathbf{n}\setminus\{i,j\}$, then*

$$E_{ij;\mathbf{a}} \sim_2 e_{ij;a_ia_j}e_{j_1j;a_{j_1}1}\cdots a_{j_{n-2}j;a_{j_{n-2}}1}.$$ □

Lemma 5.7. *Suppose $i,j,k_1,\ldots,k_t \in \mathbf{n}$ are distinct, and let $a,b,c_1,\ldots,c_t \in M$, where $t \geq 0$. Then*

$$e_{ij;ab}e_{k_1j;c_11}\cdots e_{k_tj;c_t1} \sim_2 e_{ji;ba}e_{k_1i;c_11}\cdots e_{k_ti;c_t1}e_{ij;11}.$$

Proof. We use induction on t. If $t=0$, then the words $e_{k_1j;c_11}\cdots e_{k_tj;c_t1}$ and $e_{k_1i;c_11}\cdots e_{k_ti;c_t1}$ are empty, and we have $e_{ij;ab} \sim e_{ji;ba}e_{ij;11}$, by $(\mathrm{R1})_2$. If $t \geq 1$, then

$$\begin{aligned} e_{ij;ab}e_{k_1j;c_11}\cdots e_{k_{t-1}j;c_{t-1}1}e_{k_tj;c_t1} &\sim_2 e_{ji;ba}e_{k_1i;c_11}\cdots e_{k_{t-1}i;c_{t-1}1}e_{ij;11}e_{k_tj;c_t1} && \text{by an induction hypothesis} \\ &\sim_2 e_{ji;ba}e_{k_1i;c_11}\cdots e_{k_{t-1}i;c_{t-1}1}e_{ji;11}e_{k_ti;c_t1}e_{ij;11} && \text{by (R3b)}_2 \\ &\sim_2 e_{ji;ba}e_{k_1i;c_11}\cdots e_{k_{t-1}i;c_{t-1}1}e_{k_ti;c_t1}e_{ij;11} && \text{by (R3a)}_2 \text{ if } t\geq 2\text{, or (R1)}_2 \text{ if } t=1, \end{aligned}$$

completing the proof. □

Lemma 5.8. *Let $i,j,k \in \mathbf{n}$ with $i \neq j$ and $k \neq j$, and let $a,b,c,d \in M$. Then*

(i) $e_{ij;ab}e_{kj;cd} \sim_2 e_{ij;ab}e_{kj;c1}$;

(ii) $e_{ij;ab}e_{kj;c1}e_{kj;d1} \sim_2 e_{ij;ab}e_{kj;cd,1}$.

Proof. For (i), we have

$$\begin{aligned}
e_{ij;ab}e_{kj;cd} &\sim_2 e_{ij;ab}e_{kj;11}e_{kj;cd} && \text{by (R1)}_2 \text{ or (R3a)}_2 \\
&\sim_2 e_{ij;ab}e_{kj;cc} && \text{by (R1)}_2 \\
&\sim_2 e_{ij;ab}e_{kj;11}e_{kj;c1} && \text{by (R1)}_2 \\
&\sim_2 e_{ij;ab}e_{kj;c1} && \text{by (R1)}_2 \text{ or (R3a)}_2.
\end{aligned}$$

For (ii), $e_{ij;ab}e_{kj;c1}e_{kj;d1} \sim_2 e_{ij;ab}e_{kj;cd,d} \sim_2 e_{ij;ab}e_{kj;cd,1}$, by $(\text{R1})_2$ and Part (i). □

Proof of Theorem 5.2. It remains to show that the relations from R_2 imply those from R_n'. The relations from R_n' all involve at least one word of the form $E_{ij;\mathbf{a}}$. Recall that if $a_k = 1$ for all $k \in \mathbf{n} \setminus \{i,j\}$, the word $E_{ij;\mathbf{a}}$ is simply defined to be $e_{ij;a_ia_j}$. But in this case, if $\mathbf{n} \setminus \{i,j\} = \{i_1, \ldots, i_{n-2}\}$, with $i_1 < \cdots < i_{n-2}$, then

$$E_{ij;\mathbf{a}} = e_{ij;a_ia_j} \sim_2 e_{ij;a_ia_j}e_{i_1j;11}\cdots e_{i_{n-2}j;11} = e_{ij;a_ia_j}e_{i_1j;a_{i_1}1}\cdots e_{i_{n-2}j;a_{i_{n-2}}1},$$

by $(\text{R3a})_2$. So, for uniformity, it will be convenient to assume that $E_{ij;\mathbf{a}}$ is always given by the longer expression.

$(\text{R1})_n'$: Here we have $E_{ij;\mathbf{a}} = e_{ij;a_ia_j}e_{i_1j;a_{i_1}1}\cdots e_{i_{n-2}j;a_{i_{n-2}}1} \sim_2 e_{ij;a_ia_j}e_{i_1j;a_{i_1}1}\cdots e_{i_{n-2}j;a_{i_{n-2}}1}e_{ij;11} = E_{ij;\mathbf{a}}e_{ij}$, by $(\text{R3a})_2$, and $E_{ij;\mathbf{a}} = e_{ij;a_ia_j}e_{i_1j;a_{i_1}1}\cdots e_{i_{n-2}j;a_{i_{n-2}}1} \sim_2 e_{ji;a_ja_i}e_{i_1i;a_{i_1}1}\cdots e_{i_{n-2}i;a_{i_{n-2}}1}e_{ij;11} = E_{ji;\mathbf{a}}e_{ij}$, by Lemma 5.7.

Now that we know $(\text{R1})_n'$ holds (modulo R_2), we can quickly derive $(\text{R3})_n'$, $(\text{R5})_n'$ and $(\text{R6})_n'$.

$(\text{R3})_n'$: Here we have $E_{ik:\mathbf{a}}e_{jk} \sim_2 E_{ik:\mathbf{a}}e_{ik}e_{jk} \sim E_{ik:\mathbf{a}}e_{ik} \sim E_{ik:\mathbf{a}}$, using $(\text{R1})_n'$ and $(\text{R3a})_2$.

$(\text{R5})_n'$: Here we have $E_{ki;\mathbf{a}}e_{ij}e_{jk} \sim_2 E_{ki;\mathbf{a}}e_{ki}e_{ij}e_{jk} \sim_2 E_{ki;\mathbf{a}}e_{ik}e_{kj}e_{ji}e_{ik} \sim_2 E_{ik;\mathbf{a}}e_{kj}e_{ji}e_{ik}$, by $(\text{R1})_n'$ and $(\text{R5})_2$.

$(\text{R6})_n'$: This is almost identical to $(\text{R5})_n'$.

$(\text{R2})_n'$: Relabelling the elements of $\mathbf{n}$, if necessary, and using Corollary 5.6, we may assume that $(i,j,k,l) = (1,2,3,4)$, and it suffices to show that $E_{12;\mathbf{a}}e_{34} \sim_2 E_{34;\mathbf{a}}e_{12}$. Here we have

$$\begin{aligned}
E_{12;\mathbf{a}}e_{34} &= e_{12;a_1a_2}e_{32;a_31}e_{42;a_41}\cdot e_{52;a_51}\cdots e_{n2;a_n1}\cdot e_{34;11} \\
&\sim_2 e_{12;a_1a_2}e_{32;a_31}e_{42;a_41}e_{34;11}\cdot e_{52;a_51}\cdots e_{n2;a_n1} && \text{by (R2)}_2 \\
&\sim_2 e_{12;a_1a_2}e_{32;a_31}e_{42;a_41}e_{34;11}e_{14;11}e_{14;11}\cdot e_{52;a_51}\cdots e_{n2;a_n1} && \text{by (R3a)}_2 \\
&\sim_2 e_{12;a_1a_2}e_{32;a_31}e_{34;1a_4}e_{32;11}e_{14;11}e_{14;11}\cdot e_{52;a_51}\cdots e_{n2;a_n1} && \text{by (R4b)}_2 \\
&\sim_2 e_{12;a_1a_2}e_{34;a_3a_4}e_{32;11}e_{32;11}e_{14;11}e_{14;11}\cdot e_{52;a_51}\cdots e_{n2;a_n1} && \text{by (R4a)}_2 \\
&\sim_2 e_{34;a_3a_4}e_{12;a_1a_2}e_{32;11}e_{32;11}e_{14;11}e_{14;11}\cdot e_{52;a_51}\cdots e_{n2;a_n1} && \text{by (R2)}_2 \\
&\sim_2 e_{34;a_3a_4}e_{12;a_1a_2}e_{14;11}e_{14;11}\cdot e_{52;a_51}\cdots e_{n2;a_n1} && \text{by (R3a)}_2 \\
&\sim_2 e_{34;a_3a_4}e_{14;a_11}e_{12;1a_2}e_{14;11}\cdot e_{52;a_51}\cdots e_{n2;a_n1} && \text{by (R4a)}_2 \\
&\sim_2 e_{34;a_3a_4}e_{14;a_11}e_{24;a_21}e_{12;11}\cdot e_{52;a_51}\cdots e_{n2;a_n1} && \text{by (R4a)}_2 \\
&\sim_2 e_{34;a_3a_4}e_{14;a_11}e_{24;a_21}e_{12;11}\cdot e_{42;11}\cdot e_{52;a_51}\cdots e_{n2;a_n1} && \text{by (R3a)}_2 \\
&\sim_2 e_{34;a_3a_4}e_{14;a_11}e_{24;a_21}e_{12;11}\cdot e_{24;11}\cdot e_{54;a_51}\cdots e_{n4;a_n1}\cdot e_{42;11} && \text{by Lemma 5.7} \\
&\sim_2 e_{34;a_3a_4}e_{14;a_11}e_{24;a_21}e_{34;11}e_{12;11}\cdot e_{24;11}\cdot e_{54;a_51}\cdots e_{n4;a_n1}\cdot e_{42;11} && \text{by (R3a)}_2 \\
&\sim_2 e_{34;a_3a_4}e_{14;a_11}e_{24;a_21}e_{12;11}e_{34;11}\cdot e_{24;11}\cdot e_{54;a_51}\cdots e_{n4;a_n1}\cdot e_{42;11} && \text{by (R2)}_2 \\
&\sim_2 e_{34;a_3a_4}e_{14;a_11}e_{24;a_21}e_{12;11}e_{34;11}\cdot e_{54;a_51}\cdots e_{n4;a_n1}\cdot e_{42;11} && \text{by (R3a)}_2 \\
&\sim_2 e_{34;a_3a_4}e_{14;a_11}e_{24;a_21}e_{34;11}e_{12;11}\cdot e_{54;a_51}\cdots e_{n4;a_n1}\cdot e_{42;11} && \text{by (R2)}_2 \\
&\sim_2 e_{34;a_3a_4}e_{14;a_11}e_{24;a_21}e_{12;11}\cdot e_{54;a_51}\cdots e_{n4;a_n1}\cdot e_{42;11} && \text{by (R3a)}_2 \\
&\sim_2 e_{34;a_3a_4}e_{14;a_11}e_{24;a_21}\cdot e_{54;a_51}\cdots e_{n4;a_n1}\cdot e_{12;11}e_{42;11} && \text{by (R2)}_2 \\
&\sim_2 e_{34;a_3a_4}e_{14;a_11}e_{24;a_21}\cdot e_{54;a_51}\cdots e_{n4;a_n1}\cdot e_{12;11} = E_{34;\mathbf{a}}e_{12} && \text{by (R3a)}_2.
\end{aligned}$$

$(\text{R4})_n'$: As in the previous calculation, we may assume $(i,j,k) = (1,2,3)$. First,

$$E_{12;\mathbf{a}}e_{13} = e_{12;a_1a_2}e_{32;a_31}\cdot e_{42;a_41}\cdots e_{n2;a_n1}\cdot e_{13;11}$$

$$
\begin{aligned}
&\sim_2 e_{12;a_1a_2}e_{32;a_31}e_{13;11}\cdot e_{42;a_41}\cdots e_{n2;a_n1} && \text{by (R2)}_2\\
&\sim_2 e_{12;a_1a_2}e_{13;1a_3}e_{12;11}\cdot e_{42;a_41}\cdots e_{n2;a_n1} && \text{by (R4b)}_2\\
&\sim_2 e_{13;a_1a_3}e_{12;1a_2}e_{12;11}\cdot e_{42;a_41}\cdots e_{n2;a_n1} && \text{by (R4a)}_2\\
&\sim_2 e_{13;a_1a_3}e_{13;11}e_{21;a_21}e_{12;11}\cdot e_{42;a_41}\cdots e_{n2;a_n1} && \text{by (R1)}_2\text{ twice}\\
&\sim_2 e_{13;a_1a_3}e_{23;a_21}e_{21;11}e_{12;11}\cdot e_{42;a_41}\cdots e_{n2;a_n1} && \text{by (R4b)}_2\\
&\sim_2 e_{13;a_1a_3}e_{23;a_21}e_{12;11}\cdot e_{42;a_41}\cdots e_{n2;a_n1} && \text{by (R1)}_2\\
&\sim_2 e_{13;a_1a_3}e_{23;a_21}e_{12;11}\cdot e_{32;11}\cdot e_{42;a_41}\cdots e_{n2;a_n1} && \text{by (R3a)}_2\\
&\sim_2 e_{13;a_1a_3}e_{23;a_21}e_{12;11}\cdot e_{23;11}\cdot e_{43;a_41}\cdots e_{n3;a_n1}\cdot e_{32;11} && \text{by Lemma 5.7}\\
&\sim_2 e_{13;a_1a_3}e_{23;a_21}e_{43;11}e_{12;11}\cdot e_{23;11}\cdot e_{43;a_41}\cdots e_{n3;a_n1}\cdot e_{32;11} && \text{by (R3a)}_2\\
&\sim_2 e_{13;a_1a_3}e_{23;a_21}e_{12;11}e_{43;11}\cdot e_{23;11}\cdot e_{43;a_41}\cdots e_{n3;a_n1}\cdot e_{32;11} && \text{by (R2)}_2\\
&\sim_2 e_{13;a_1a_3}e_{23;a_21}e_{12;11}e_{43;11}\cdot e_{43;a_41}\cdots e_{n3;a_n1}\cdot e_{32;11} && \text{by (R3a)}_2\\
&\sim_2 e_{13;a_1a_3}e_{23;a_21}e_{43;11}e_{12;11}\cdot e_{43;a_41}\cdots e_{n3;a_n1}\cdot e_{32;11} && \text{by (R2)}_2\\
&\sim_2 e_{13;a_1a_3}e_{23;a_21}e_{12;11}\cdot e_{43;a_41}\cdots e_{n3;a_n1}\cdot e_{32;11} && \text{by (R3a)}_2\\
&\sim_2 e_{13;a_1a_3}e_{23;a_21}\cdot e_{43;a_41}\cdots e_{n3;a_n1}\cdot e_{12;11}e_{32;11} && \text{by (R2)}_2\\
&\sim_2 e_{13;a_1a_3}e_{23;a_21}\cdot e_{43;a_41}\cdots e_{n3;a_n1}\cdot e_{12;11} = E_{13;\mathbf{a}}e_{12} && \text{by (R3a)}_2,
\end{aligned}
$$

establishing the first part of $(\mathrm{R4})'_n$. For the second part,

$$
\begin{aligned}
E_{13;\mathbf{a}}e_{12} &= e_{13;a_1a_3}e_{23;a_21}\cdot e_{43;a_41}\cdots e_{n3;a_n1}\cdot e_{12;11}\\
&\sim_2 e_{13;a_1a_3}e_{23;a_21}e_{12;11}\cdot e_{43;a_41}\cdots e_{n3;a_n1} && \text{by (R2)}_2\\
&\sim_2 e_{13;a_1a_3}e_{12;1a_2}e_{13;11}\cdot e_{43;a_41}\cdots e_{n3;a_n1} && \text{by (R4b)}_2\\
&\sim_2 e_{23;a_2a_3}e_{12;a_11}e_{13;11}\cdot e_{43;a_41}\cdots e_{n3;a_n1} && \text{by (R4b)}_2\\
&\sim_2 e_{23;a_2a_3}e_{13;a_11}e_{12;11}\cdot e_{43;a_41}\cdots e_{n3;a_n1} && \text{by (R4a)}_2\\
&\sim_2 e_{23;a_2a_3}e_{13;a_11}\cdot e_{43;a_41}\cdots e_{n3;a_n1}\cdot e_{12;11} = E_{23;\mathbf{a}}e_{12} && \text{by (R2)}_2.
\end{aligned}
$$

$(\mathrm{R7})'_n$: Here we may assume that $(i,j)=(1,2)$, but (because i,j,k,l are not necessarily distinct), there are several possibilities for the values of k,l:

(i) $(k,l)=(1,2)$; (ii) $(k,l)=(2,1)$; (iii) $(k,l)=(1,3)$; (iv) $(k,l)=(3,1)$; (v) $(k,l)=(3,2)$; (vi) $(k,l)=(2,3)$; (vii) $(k,l)=(3,4)$.

In each case, we have $\mathbf{c}=(c_1,\ldots,c_n)=\mathbf{a}(\varepsilon_{12}\cdot\mathbf{b})=(a_1b_1,a_2b_1,a_3b_3,\ldots,a_nb_n)$. In Case (i), we have

$$
\begin{aligned}
E_{12;\mathbf{a}}E_{12;\mathbf{b}} &= e_{12;a_1a_2}e_{32;a_31}\cdots e_{n2;a_n1}e_{12;b_1b_2}e_{32;b_31}\cdots e_{n2;b_n1}\\
&\sim_2 e_{12;a_1a_2}e_{32;a_31}\cdots e_{n2;a_n1}e_{12;b_11}e_{32;b_31}\cdots e_{n2;b_n1} && \text{by Lemma 5.8(i)}\\
&\sim_2 e_{12;a_1a_2}e_{12;b_11}(e_{32;a_31}e_{32;b_31})\cdots(e_{n2;a_n1}e_{n2;b_n1}) && \text{by Corollary 5.5, repeatedly}\\
&\sim_2 e_{12;a_1b_1,a_2b_1}e_{32;a_3b_3,1}\cdots e_{n2;a_nb_n,1} && \text{by (R1)}_2\text{ and Lemma 5.8(ii)}\\
&\sim_2 e_{12;a_1b_1,a_2b_1}e_{32;a_3b_3,1}\cdots e_{n2;a_nb_n,1}e_{12;11} = E_{12;\mathbf{c}}e_{12} && \text{by (R3a)}_2,
\end{aligned}
$$

For Case (ii), we use $(\mathrm{R1})'_n$, Case (i) of $(\mathrm{R7})'_n$ and $(\mathrm{R1})_2$:

$$E_{12;\mathbf{a}}E_{21;\mathbf{b}} \sim_2 E_{12;\mathbf{a}}E_{12;\mathbf{b}}e_{21} \sim_2 E_{12;\mathbf{c}}e_{12}e_{21} \sim_2 E_{12;\mathbf{c}}e_{21}.$$

In Case (iii), we have

$$
\begin{aligned}
E_{12;\mathbf{a}}E_{13;\mathbf{b}} &= e_{12;a_1a_2}e_{32;a_31}e_{42;a_41}\cdots e_{n2;a_n1}e_{13;b_1b_3}e_{23;b_21}e_{43;b_41}\cdots e_{n3;b_n1}\\
&\sim_2 e_{12;a_1a_2}e_{32;a_31}e_{13;b_1b_3}e_{42;a_41}\cdots e_{n2;a_n1}e_{23;b_21}e_{43;b_41}\cdots e_{n3;b_n1} && \text{by (R2)}_2\\
&\sim_2 e_{12;a_1a_2}e_{32;a_31}e_{13;b_1b_3}e_{42;a_41}\cdots e_{n2;a_n1}e_{32;1b_2}e_{42;b_41}\cdots e_{n2;b_n1}e_{23;11} && \text{by Lemma 5.7}\\
&\sim_2 e_{12;a_1a_2}e_{12;b_1b_3}e_{13;1,a_3b_3}e_{42;a_41}\cdots e_{n2;a_n1}e_{32;1b_2}e_{42;b_41}\cdots e_{n2;b_n1}e_{23;11} && \text{by (R4b)}_2\\
&\sim_2 e_{12;a_1b_1,a_2b_1}e_{13;1,a_3b_3}e_{42;a_41}\cdots e_{n2;a_n1}e_{32;1b_2}e_{42;b_41}\cdots e_{n2;b_n1}e_{23;11} && \text{by (R1)}_2\\
&\sim_2 e_{12;a_1b_1,a_2b_1}e_{13;1,a_3b_3}e_{42;a_41}\cdots e_{n2;a_n1}e_{42;b_41}\cdots e_{n2;b_n1}e_{23;11} && \text{by (R3a)}_2\text{ unless } n=3 \qquad (*)
\end{aligned}
$$

$$
\begin{aligned}
&\sim_2 e_{12;a_1b_1,a_2b_1} e_{42;a_41} \cdots e_{n2;a_n1} e_{42;b_41} \cdots e_{n2;b_n1} e_{13;1,a_3b_3} e_{23;11} && \text{by } (\mathrm{R2})_2 \\
&\sim_2 e_{12;a_1b_1,a_2b_1} (e_{42;a_41} e_{42;b_41}) \cdots (e_{n2;a_n1} e_{n2;b_n1}) e_{13;1,a_3b_3} e_{23;11} && \text{by Corollary 5.5} \\
&\sim_2 e_{12;a_1b_1,a_2b_1} e_{42;a_4b_4,1} \cdots e_{n2;a_nb_n,1} e_{13;1,a_3b_3} e_{23;11} && \text{by Lemma 5.8(ii)} \\
&\sim_2 e_{12;a_1b_1,a_2b_1} e_{42;a_4b_4,1} \cdots e_{n2;a_nb_n,1} e_{12;11} e_{13;1,a_3b_3} e_{23;11} && \text{by } (\mathrm{R3a})_2 \\
&\sim_2 e_{12;a_1b_1,a_2b_1} e_{42;a_4b_4,1} \cdots e_{n2;a_nb_n,1} e_{32;a_3b_3,1} e_{13;11} e_{23;11} && \text{by } (\mathrm{R4b})_2 \\
&\sim_2 e_{12;a_1b_1,a_2b_1} e_{32;a_3b_3,1} e_{42;a_4b_4,1} \cdots e_{n2;a_nb_n,1} e_{13;11} = E_{12;\mathbf{c}} e_{13} && \text{by Corollary 5.5 and } (\mathrm{R3a})_2.
\end{aligned}
$$

Note that at the step labelled $(*)$, relation $(\mathrm{R3a})_2$ does not apply if $n = 3$, since then the word $e_{42;a_41} \cdots e_{n2;a_n1}$ is empty. However, this step can still be accomplished, albeit by using more relations:

$$
\begin{aligned}
e_{12;a_1b_1,a_2b_1} e_{13;1,a_3b_3} e_{42;a_41} \cdots e_{n2;a_n1} e_{32;1b_2} &= e_{12;a_1b_1,a_2b_1} e_{13;1,a_3b_3} e_{32;1b_2} \\
&\sim_2 e_{13;a_1b_1,a_3b_3} e_{12;1,a_2b_1} e_{32;1b_2} && \text{by } (\mathrm{R4a})_2 \\
&\sim_2 e_{13;a_1b_1,a_3b_3} e_{12;1,a_2b_1} && \text{by } (\mathrm{R3a})_2 \\
&\sim_2 e_{12;a_1b_1,a_2b_1} e_{13;1,a_3b_3} && \text{by } (\mathrm{R4a})_2 \\
&= e_{12;a_1b_1,a_2b_1} e_{13;1,a_3b_3} e_{42;a_41} \cdots e_{n2;a_n1}.
\end{aligned}
$$

Again, we may deduce Case (iv) from Case (iii), together with $(\mathrm{R1})'_n$ and $(\mathrm{R1})_2$:

$$
E_{12;\mathbf{a}} E_{31;\mathbf{b}} \sim_2 E_{12;\mathbf{a}} E_{13;\mathbf{b}} e_{31} \sim_2 E_{12;\mathbf{c}} e_{13} e_{31} \sim_2 E_{12;\mathbf{c}} e_{31}.
$$

In Case (v), we have

$$
\begin{aligned}
E_{12;\mathbf{a}} E_{32;\mathbf{b}} &= e_{12;a_1a_2} e_{32;a_31} e_{42;a_41} \cdots e_{n2;a_n1} e_{32;b_3b_2} e_{12;b_11} e_{42;b_41} \cdots e_{n2;b_n1} \\
&\sim_2 e_{12;a_1a_2} e_{32;a_31} e_{42;a_41} \cdots e_{n2;a_n1} e_{32;b_31} e_{12;b_11} e_{42;b_41} \cdots e_{n2;b_n1} && \text{by Lemma 5.8(i)} \\
&\sim_2 e_{12;a_1a_2} e_{12;b_11} (e_{32;a_31} e_{32;b_31})(e_{42;a_41} e_{42;b_41}) \cdots (e_{n2;a_n1} e_{n2;b_n1}) && \text{by Corollary 5.5} \\
&\sim_2 e_{12;a_1b_1,a_2b_1} e_{32;a_3b_3,1} e_{42;a_4b_4,1} \cdots e_{n2;a_nb_n,1} && \text{by } (\mathrm{R1})_2 \text{ and Lemma 5.8(ii)} \\
&\sim_2 e_{12;a_1b_1,a_2b_1} e_{32;a_3b_3,1} e_{42;a_4b_4,1} \cdots e_{n2;a_nb_n,1} e_{32;11} = E_{12;\mathbf{c}} e_{32} && \text{by } (\mathrm{R3a})_2.
\end{aligned}
$$

Case (vi) follows quickly from (v), $(\mathrm{R1})'_n$ and $(\mathrm{R1})_2$. Finally, for Case (vii), we first observe that for any $u, v, x, y \in M$,

$$
\begin{aligned}
e_{12;uv} e_{32;x1} e_{42;y1} e_{34;11} &\sim_2 e_{12;uv} e_{32;x1} e_{32;11} e_{34;1y} && \text{by } (\mathrm{R4b})_2 \\
&\sim_2 e_{12;uv} e_{32;x1} e_{34;1y} && \text{by } (\mathrm{R1})_2 \\
&\sim_2 e_{12;uv} e_{34;xy} e_{32;11} && \text{by } (\mathrm{R4a})_2 \\
&\sim_2 e_{34;xy} e_{12;uv} e_{32;11} && \text{by } (\mathrm{R2})_2 \\
&\sim_2 e_{34;xy} e_{12;uv} && \text{by } (\mathrm{R3a})_2 \\
&\sim_2 e_{12;uv} e_{34;xy} && \text{by } (\mathrm{R2})_2.
\end{aligned}
$$

We then calculate

$$
\begin{aligned}
E_{12;\mathbf{a}} E_{34;\mathbf{b}} &= e_{12;a_1a_2} e_{32;a_31} e_{42;a_41} e_{52;a_51} \cdots e_{n2;a_n1} e_{34;b_3b_4} e_{14;b_11} e_{24;b_21} e_{54;b_51} \cdots e_{n4;b_n1} \\
&\sim_2 e_{12;a_1a_2} e_{32;a_31} e_{42;a_41} e_{34;b_3b_4} e_{14;b_11} e_{52;a_51} \cdots e_{n2;a_n1} e_{24;b_21} e_{54;b_51} \cdots e_{n4;b_n1} && \text{by } (\mathrm{R2})_2 \\
&\sim_2 e_{12;a_1a_2} e_{32;a_31} e_{42;a_41} e_{34;b_3b_4} e_{14;b_11} e_{52;a_51} \cdots e_{n2;a_n1} e_{42;1b_2} e_{52;b_51} \cdots e_{n2;b_n1} e_{24;11} && \text{by Lemma 5.7} \\
&\sim_2 e_{12;a_1a_2} e_{32;a_31} e_{34;b_3,a_4b_4} e_{32;1b_4} e_{14;b_11} e_{52;a_51} \cdots e_{n2;a_n1} e_{42;1b_2} e_{52;b_51} \cdots e_{n2;b_n1} e_{24;11} && \text{by } (\mathrm{R4b})_2 \\
&\sim_2 e_{12;a_1a_2} e_{34;a_3b_3,a_4b_4} e_{32;1,b_3} e_{14;b_11} e_{32;1b_4} e_{52;a_51} \cdots e_{n2;a_n1} e_{42;1b_2} e_{52;b_51} \cdots e_{n2;b_n1} e_{24;11} && \text{by } (\mathrm{R2})_2 \text{ and } (\mathrm{R4a})_2 \\
&\sim_2 e_{34;a_3b_3,a_4b_4} e_{12;a_1a_2} e_{32;1,b_3} e_{14;b_11} e_{32;1b_4} e_{52;a_51} \cdots e_{n2;a_n1} e_{42;1b_2} e_{52;b_51} \cdots e_{n2;b_n1} e_{24;11} && \text{by } (\mathrm{R2})_2 \\
&\sim_2 e_{34;a_3b_3,a_4b_4} e_{12;a_1a_2} e_{14;b_11} e_{32;1b_4} e_{52;a_51} \cdots e_{n2;a_n1} e_{42;1b_2} e_{52;b_51} \cdots e_{n2;b_n1} e_{24;11} && \text{by } (\mathrm{R3a})_2 \\
&\sim_2 e_{34;a_3b_3,a_4b_4} e_{14;a_1b_1,1} e_{12;1,a_2b_1} e_{32;1b_4} e_{52;a_51} \cdots e_{n2;a_n1} e_{42;1b_2} e_{52;b_51} \cdots e_{n2;b_n1} e_{24;11} && \text{by } (\mathrm{R4a})_2 \\
&\sim_2 e_{34;a_3b_3,a_4b_4} e_{14;a_1b_1,1} e_{12;1,a_2b_1} e_{52;a_51} \cdots e_{n2;a_n1} e_{52;b_51} \cdots e_{n2;b_n1} e_{24;11} && \text{by } (\mathrm{R3a})_2 \text{ twice} \\
&\sim_2 e_{34;a_3b_3,a_4b_4} e_{12;a_1b_1,a_2b_1} e_{14;11} e_{52;a_51} \cdots e_{n2;a_n1} e_{52;b_51} \cdots e_{n2;b_n1} e_{24;11} && \text{by } (\mathrm{R4a})_2 \\
&\sim_2 e_{12;a_1b_1,a_2b_1} e_{34;a_3b_3,a_4b_4} e_{14;11} e_{52;a_51} \cdots e_{n2;a_n1} e_{52;b_51} \cdots e_{n2;b_n1} e_{24;11} && \text{by } (\mathrm{R2})_2 \\
&\sim_2 e_{12;a_1b_1,a_2b_1} e_{34;a_3b_3,a_4b_4} e_{52;a_51} \cdots e_{n2;a_n1} e_{52;b_51} \cdots e_{n2;b_n1} e_{24;11} && \text{by } (\mathrm{R3a})_2
\end{aligned}
$$

$$
\begin{array}{lr}
\sim_2 e_{34;a_3b_3,a_4b_4}e_{12;a_1b_1,a_2b_1}e_{52;a_51}\cdots e_{n2;a_n1}e_{52;b_51}\cdots e_{n2;b_n1}e_{24;11} & \text{by } (\text{R2})_2 \\
\sim_2 e_{34;a_3b_3,a_4b_4}e_{12;a_1b_1,a_2b_1}(e_{52;a_51}e_{52;b_51})\cdots(e_{n2;a_n1}e_{n2;b_n1})e_{24;11} & \text{by Corollary 5.5} \\
\sim_2 e_{34;a_3b_3,a_4b_4}e_{12;a_1b_1,a_2b_1}e_{52;a_5b_5,1}\cdots e_{n2;a_nb_n,1}e_{24;11} & \text{by Lemma 5.8(ii)} \\
\sim_2 e_{12;a_1b_1,a_2b_1}e_{52;a_5b_5,1}\cdots e_{n2;a_nb_n,1}e_{34;a_3b_3,a_4b_4}e_{24;11} & \text{by } (\text{R2})_2 \\
\sim_2 e_{12;a_1b_1,a_2b_1}e_{52;a_5b_5,1}\cdots e_{n2;a_nb_n,1}e_{34;a_3b_3,a_4b_4} & \text{by } (\text{R3a})_2 \\
\sim_2 e_{12;a_1b_1,a_2b_1}e_{34;a_3b_3,a_4b_4}e_{52;a_5b_5,1}\cdots e_{n2;a_nb_n,1} & \text{by } (\text{R2})_2 \\
\sim_2 e_{12;a_1b_1,a_2b_1}e_{32;a_3b_3,1}e_{42;a_4b_4,1}e_{34;11}e_{52;a_5b_5,1}\cdots e_{n2;a_nb_n,1} & \text{by the observation} \\
\sim_2 e_{12;a_1b_1,a_2b_1}e_{32;a_3b_3,1}e_{42;a_4b_4,1}e_{52;a_5b_5,1}\cdots e_{n2;a_nb_n,1}e_{34;11} = E_{12;\mathbf{c}}e_{34} & \text{by } (\text{R2})_2.
\end{array}
$$

We have finally shown that all the relations from R_n' may be removed, and the proof is therefore complete. □

5.2 An idempotent generated presentation for $M \wr \mathrm{Sing}_n$ with $M/\mathscr{L}$ a chain

For the duration of this section, we fix a monoid M for which $M/\mathscr{L}$ is a chain. Recall from Theorem 4.7 that, for such a monoid M, the singular wreath product $M \wr \mathrm{Sing}_n$ is generated by its idempotents: indeed, by the idempotents from the set $\mathcal{X}_1$. It is therefore our goal in this section to obtain a presentation (see Theorem 5.9) for $M \wr \mathrm{Sing}_n$ in terms of the idempotent generating set $\mathcal{X}_1$. In the special case that M is a group, some of the relations take on a simpler form (see Theorem 5.12).

With the stated goal in mind, define an alphabet

$$X_1 = \{e_{ij;a} : i,j \in \mathbf{n},\ i \neq j,\ a \in M\},$$

an epimorphism

$$\phi_1 : X_1^+ \to M \wr \mathrm{Sing}_n : e_{ij;a} \mapsto \varepsilon_{ij;a},$$

and let R_1 be the set of relations

$$
\begin{array}{rcll}
e_{ij;a}e_{ij;b} &=& e_{ij;a} & \text{for } a,b \in M \text{ and distinct } i,j \quad (\text{R1a})_1 \\
e_{ij;1}e_{ji;a}e_{ij;b} &=& e_{ji;1}e_{ij;ab} & \text{for } a,b \in M \text{ and distinct } i,j \quad (\text{R1b})_1 \\
e_{ji;a}e_{ij;c} &=& e_{ji;b}e_{ij;c} & \text{for } a,b,c \in M \text{ and distinct } i,j \text{ with } ac = bc \quad (\text{R1c})_1 \\
e_{ij;b}e_{ji;c}e_{ij;1} &=& e_{ji;a}e_{ij;bc} & \text{for } a,b,c \in M \text{ and distinct } i,j \text{ with } abc = c \quad (\text{R1d})_1 \\
e_{ji;1}e_{ij;1} &=& e_{ij;1} & \text{for distinct } i,j \quad (\text{R1e})_1 \\
e_{ij;a}e_{kl;b} &=& e_{kl;b}e_{ij;a} & \text{for } a,b \in M \text{ and distinct } i,j,k,l \quad (\text{R2})_1 \\
e_{ik;a}e_{jk;b} &=& e_{ik;a} & \text{for } a,b \in M \text{ and distinct } i,j,k \quad (\text{R3a})_1 \\
e_{ij;1}e_{jk;a}e_{kj;1} &=& e_{ji;1}e_{ik;a}e_{ki;1}e_{ij;1} & \text{for } a \in M \text{ and distinct } i,j,k \quad (\text{R3b})_1 \\
e_{ij;1}e_{ji;a}e_{ik;b} &=& e_{ji;1}e_{ik;b}e_{kj;a}e_{jk;1} & \text{for } a,b \in M \text{ and distinct } i,j,k \quad (\text{R3c})_1 \\
e_{ij;b}e_{ik;ab} = e_{ik;ab}e_{ij;b} &=& e_{jk;a}e_{ij;b} & \text{for } a,b \in M \text{ and distinct } i,j,k \quad (\text{R4})_1 \\
e_{ki}e_{ij}e_{jk} &=& e_{ik}e_{kj}e_{ji}e_{ik} & \text{for distinct } i,j,k \quad (\text{R5})_1 \\
e_{ki}e_{ij}e_{jk}e_{kl} &=& e_{ik}e_{kl}e_{li}e_{ij}e_{jl} & \text{for distinct } i,j,k,l. \quad (\text{R6})_1
\end{array}
$$

Again, we have identified X with a subset of X_1 in relations $(\text{R5})_1$ and $(\text{R6})_1$. It is important to note that some relations (such as $(\text{R1b})_1$) involve a letter of the form $e_{ij;ab}$ from X_1, where "ab" denotes a single subscript (the product of a and b in M): in particular, $e_{ij;ab}$ does not represent the letter from X_2 where a and b are separate subscripts. In order to avoid any potential confusion, we will write *all* letters from X_2 as $e_{ij;a,b}$ (not just those for which the monoid subscripts are expressed as products) for the duration of this section. We may now state the main result of this section.

Theorem 5.9. *If $M/\mathscr{L}$ is a chain, then the semigroup $M \wr \mathrm{Sing}_n$ has presentation $\langle X_1 : R_1 \rangle$ via ϕ_1.*

To prove Theorem 5.9, we begin with the presentation $\langle X_2 : R_2 \rangle$ from Theorem 5.2. Again, we think of X_1 as a subset of X_2 by identifying $e_{ij;a} \in X_1$ with $e_{ij;1,a} \in X_2$. Since $u\phi_2 = v\phi_2$ for all $(u,v) \in R_1$, as may easily be checked diagrammatically, we may add relations R_1 to the presentation to obtain $\langle X_2 : R_1 \cup R_2\rangle$. Now write $\sim_1 = R_1^\sharp$ for the congruence on X_1^+ generated by R_1.

Since $M/\mathscr{L}$ is a chain, we may fix some subset $\Omega \subseteq M\times M$ such that the following conditions are satisfied:

(i) for all $a, b \in M$, Ω contains exactly one of (a, b) or (b, a); and

(ii) for all $(a, b) \in \Omega$, $a \leq_{\mathscr{L}} b$.

Note that Ω (regarded as a binary relation) is reflexive and anti-symmetric, but need not be transitive (though Ω could be chosen to have this additional property, in which case Ω would then be a total order on M that refines the preorder $\leq_{\mathscr{L}}$). For each $(a, b) \in \Omega$, we choose some $x_{ab} \in M$ such that $a = x_{ab}b$. For each $i, j \in \mathbf{n}$ with $i \neq j$, and for each $a, b \in M$, we define the word

$$E_{ij;ab} = \begin{cases} e_{ji;x_{ab}}e_{ij;b} & \text{if } (a, b) \in \Omega \\ e_{ij;x_{ba}}e_{ji;a}e_{ij;1} & \text{if } (a, b) \notin \Omega. \end{cases}$$

(Since there is no chance of confusion, we will not generally need to write $E_{ij;a,b}$ for these words.) We begin with a simple lemma.

Lemma 5.10. *Let $i, j \in \mathbf{n}$ with $i \neq j$, and let $a, b \in M$. Then*

(i) $E_{ij;ab}\phi_2 = e_{ij;a,b}\phi_2$;

(ii) $E_{ij;ab} \sim_1 e_{ji;x}e_{ij;b}$ *for any $x \in M$ with $a = xb$;*

(iii) $E_{ij;ab} \sim_1 e_{ij;x}e_{ji;a}e_{ij;1}$ *for any $x \in M$ with $b = xa$;*

(iv) $E_{ij;1a} \sim_1 e_{ij;a}$.

Proof. Part (i) is easily checked diagrammatically. For (ii), we must consider two cases. If $(a, b) \in \Omega$, then $(\mathrm{R1c})_1$ gives $E_{ij;ab} = e_{ji;x_{ab}}e_{ij;b} \sim_1 e_{ji;x}e_{ij;b}$, since $xb = a = x_{ab}b$. If $(a, b) \notin \Omega$, then $(b, a) \in \Omega$, $b = x_{ba}a$ and $a = xb = xx_{ba}a$, so $(\mathrm{R1d})_1$ gives $E_{ij;ab} = e_{ij;x_{ba}}e_{ji;a}e_{ij;1} \sim_1 e_{ji;x}e_{ij;x_{ba}a} = e_{ji;x}e_{ij;b}$.

For (iii), if $(a, b) \in \Omega$, then $a = x_{ab}b = x_{ab}xa$, and $(\mathrm{R1d})_1$ gives $E_{ij;ab} = e_{ji;x_{ab}}e_{ij;b} = e_{ji;x_{ab}}e_{ij;xa} \sim_1 e_{ij;x}e_{ji;a}e_{ij;1}$. If $(a, b) \notin \Omega$, then $(\mathrm{R1c})_1$ gives $E_{ij;ab} = e_{ij;x_{ba}}e_{ji;a}e_{ij;1} \sim_1 e_{ij;x}e_{ji;a}e_{ij;1}$.

For (iv), we have $E_{ij;1a} \sim_1 e_{ij;a}e_{ji;1}e_{ij;1} \sim_1 e_{ij;a}e_{ij;1} \sim_1 e_{ij;a}$, using Part (iii) and then $(\mathrm{R1e})_1$ and $(\mathrm{R1a})_1$. □

By Lemma 5.10(i) and Theorem 5.2, it follows that $e_{ij;a,b} \sim_2 E_{ij;ab}$ for each i, j, a, b, so we may remove each generator $e_{ij;a,b} \in X_2 \setminus X_1$ from the presentation $\langle X_2 : R_1 \cup R_2\rangle$, replacing every occurence of such a generator in the relations by the word $E_{ij;ab} \in X_1^+$. The only relations that are affected in this way are those from R_2, excluding $(\mathrm{R5})_2$ and $(\mathrm{R6})_2$, which are just $(\mathrm{R5})_1$ and $(\mathrm{R6})_1$. We denote the relations modified in this way by:

$$E_{ij;ab}E_{ij;cd} = E_{ij;ac,bc} = E_{ji;ba}E_{ij;dc} \quad \text{for } a, b, c, d \in M \text{ and distinct } i, j \quad (\mathrm{R1})_2'$$

$$E_{ij;ab}E_{kl;cd} = E_{kl;cd}E_{ij;ab} \quad \text{for } a, b, c, d \in M \text{ and distinct } i, j, k, l \quad (\mathrm{R2})_2'$$

$$E_{ik;ab}E_{jk;1c} = E_{ik;ab} \quad \text{for } a, b, c \in M \text{ and distinct } i, j, k \quad (\mathrm{R3a})_2'$$

$$E_{ik;ab}E_{jk;c1} = E_{ki;ba}E_{ji;c1}E_{ik;11} \quad \text{for } a, b, c \in M \text{ and distinct } i, j, k \quad (\mathrm{R3b})_2'$$

$$E_{ik;aa}E_{jk;b1} = E_{ik;11}E_{jk;b1}E_{ik;a1} \quad \text{for } a, b \in M \text{ and distinct } i, j, k \quad (\mathrm{R3c})_2'$$

$$E_{ij;ab}E_{ik;cd} = E_{ik;ac,d}E_{ij;1,bc} = E_{jk;bc,d}E_{ij;ac,1} \quad \text{for } a, b, c, d \in M \text{ and distinct } i, j, k \quad (\mathrm{R4a})_2'$$

$$E_{ij;c,ad}E_{ik;1,bd} = E_{ik;c,bd}E_{ij;1,ad} = E_{jk;ab}E_{ij;cd} \quad \text{for } a, b, c, d \in M \text{ and distinct } i, j, k, \quad (\mathrm{R4b})_2'$$

and denote the whole set of relations modified in this way by R_2'. So the presentation has now become $\langle X_1 : R_1 \cup R_2'\rangle$, and we must show that the relations from R_2' may be removed. We now pause to prove a technical lemma that will be useful on a number of occasions. For its proof, and for many calculations in this section, we observe that the first part of $(\mathrm{R4})_1$ implies

$$e_{ij;a}e_{ik;b} \sim_1 e_{ik;b}e_{ij;a} \quad \text{for } a, b \in M \text{ and distinct } i, j, k,$$

since we either have $a = xb$ or $b = xa$ for some $x \in M$, as $M/\mathscr{L}$ is a chain.

Lemma 5.11. *Let $i, j, k \in \mathbf{n}$ be distinct, and let $a, b, c, d \in M$. Then*

(i) $e_{jk;a}e_{kj;b}e_{ki;c}e_{ik;d} \sim_1 e_{jk;a}e_{kj;bd}e_{ki;cd}e_{ik;1}$;

(ii) $e_{ki;1}e_{ik;b}e_{ij;ab} \sim_1 e_{ik;1}e_{ij;a}e_{ji;b}e_{ij;1}$;

(iii) $e_{ki;1}e_{ik;ab}e_{ij;b} \sim_1 e_{ik;1}e_{ji;a}e_{ij;b}$;

(iv) $e_{ki;a}e_{ik;c}e_{ij;d} \sim_1 e_{ji;b}e_{ik;c}e_{ij;d}$ *if $ac = bd$.*

Proof. For (i), we have

$$\begin{aligned}
e_{jk;a}e_{kj;b}e_{ki;c}e_{ik;d} &\sim_1 e_{jk;a}e_{ki;c}e_{kj;b}e_{ik;d} && \text{by (R4)}_1 \\
&\sim_1 e_{jk;a}e_{ik;1}e_{ki;c}e_{ik;d}e_{ij;bd} && \text{by (R3a)}_1 \text{ and (R4)}_1 \\
&\sim_1 e_{jk;a}e_{ki;1}e_{ik;cd}e_{ij;bd} && \text{by (R1b)}_1 \\
&\sim_1 e_{jk;a}e_{ik;1}e_{ki;cd}e_{ik;1}e_{ij;bd} && \text{by (R1b)}_1 \\
&\sim_1 e_{jk;a}e_{ki;cd}e_{ik;1}e_{ij;bd} && \text{by (R3a)}_1 \\
&\sim_1 e_{jk;a}e_{ki;cd}e_{kj;bd}e_{ik;1} && \text{by (R4)}_1 \\
&\sim_1 e_{jk;a}e_{kj;bd}e_{ki;cd}e_{ik;1} && \text{by (R4)}_1.
\end{aligned}$$

For (ii), we have

$$\begin{aligned}
e_{ik;1}e_{ij;a}e_{ji;b}e_{ij;1} &\sim_1 e_{kj;a}e_{ik;1}e_{ji;b}e_{ij;1} && \text{by (R4)}_1 \\
&\sim_1 e_{kj;a}e_{ji;b}e_{jk;b}e_{ij;1} && \text{by (R4)}_1 \\
&\sim_1 e_{kj;a}e_{jk;b}e_{ji;b}e_{ij;1} && \text{by (R4)}_1 \\
&\sim_1 e_{kj;a}e_{ki;1}e_{jk;b}e_{ij;1} && \text{by (R4)}_1 \\
&\sim_1 e_{ki;1}e_{kj;a}e_{ij;1}e_{ik;b} && \text{by (R4)}_1 \text{ twice} \\
&\sim_1 e_{ki;1}e_{kj;a}e_{ik;b} && \text{by (R3a)}_1 \\
&\sim_1 e_{ki;1}e_{ik;b}e_{ij;ab} && \text{by (R4)}_1.
\end{aligned}$$

For (iii), we have $e_{ki;1}e_{ik;ab}e_{ij;b} \sim_1 e_{ki;1}e_{jk;a}e_{ij;b} \sim_1 e_{jk;a}e_{ji;a}e_{ij;b} \sim_1 e_{ik;1}e_{ji;a}e_{ij;b}$, by three applications of (R4)_1. For (iv), we must consider two cases. If $c = xd$ for some $x \in M$, then

$$\begin{aligned}
e_{ki;a}e_{ik;c}e_{ij;d} = e_{ki;a}e_{ik;xd}e_{ij;d} &\sim_1 e_{ki;a}e_{jk;x}e_{ij;d} && \text{by (R4)}_1 \\
&\sim_1 e_{ji;ax}e_{jk;x}e_{ij;d} && \text{by (R4)}_1 \\
&\sim_1 e_{ji;ax}e_{ik;xd}e_{ij;d} && \text{by (R4)}_1 \\
&\sim_1 e_{ji;ax}e_{ij;d}e_{ik;xd} && \text{by (R4)}_1 \\
&\sim_1 e_{ji;b}e_{ij;d}e_{ik;c} && \text{by (R1c)}_1\text{, as } axd = ac = bd \text{ and } xd = c \\
&\sim_1 e_{ji;b}e_{ik;c}e_{ij;d} && \text{by (R4)}_1
\end{aligned}$$

while if $d = yc$ for some $y \in M$, then

$$\begin{aligned}
e_{ji;b}e_{ik;c}e_{ij;d} = e_{ji;b}e_{ik;c}e_{ij;yc} &\sim_1 e_{ji;b}e_{ij;yc}e_{ik;c} && \text{by (R4)}_1 \\
&\sim_1 e_{ji;b}e_{kj;y}e_{ik;c} && \text{by (R4)}_1 \\
&\sim_1 e_{ki;by}e_{kj;y}e_{ik;c} && \text{by (R4)}_1 \\
&\sim_1 e_{ki;by}e_{ij;yc}e_{ik;c} && \text{by (R4)}_1 \\
&\sim_1 e_{ki;by}e_{ik;c}e_{ij;yc} && \text{by (R4)}_1 \\
&\sim_1 e_{ki;a}e_{ik;c}e_{ij;d} && \text{by (R1c)}_1\text{, as } byc = bd = ac \text{ and } yc = d,
\end{aligned}$$

completing the proof. □

Proof of Theorem 5.9. We consider the relations from R_2' one at a time; each splits into several cases, depending on the relationship between the monoid subscripts in the $\leq_{\mathscr{L}}$ order.

$\text{(R1)}_2'$: There are four cases to consider:

(i) $a \leq_{\mathscr{L}} b$ and $c \leq_{\mathscr{L}} d$;

(ii) $a \leq_{\mathscr{L}} b$ and $d \leq_{\mathscr{L}} c$;

(iii) $b \leq_{\mathscr{L}} a$ and $c \leq_{\mathscr{L}} d$;

(iv) $b \leq_{\mathscr{L}} a$ and $d \leq_{\mathscr{L}} c$.

In Case (i), writing $a = xb$ and $c = yd$, we have

$$\begin{aligned}
E_{ij;ab}E_{ij;cd} &\sim_1 e_{ji;x}e_{ij;b}e_{ji;y}e_{ij;d} && \text{by Lemma 5.10(ii)}\\
&\sim_1 e_{ji;x}e_{ji;1}e_{ij;b}e_{ij;1}e_{ji;y}e_{ij;d} && \text{by (R1a)}_1 \text{ twice}\\
&\sim_1 e_{ji;x}e_{ji;1}e_{ij;b}e_{ji;1}e_{ij;c} && \text{by (R1b)}_1\text{, noting that } c = yd\\
&\sim_1 e_{ji;x}e_{ij;1}e_{ji;b}e_{ij;c} && \text{by (R1b)}_1\\
&\sim_1 e_{ji;x}e_{ji;1}e_{ij;bc} && \text{by (R1b)}_1\\
&\sim_1 e_{ji;x}e_{ij;bc} && \text{by (R1a)}_1\\
&\sim_1 E_{ij;ac,bc} && \text{by Lemma 5.10(ii), noting that } xbc = ac,
\end{aligned}$$

while

$$\begin{aligned}
E_{ji;ba}E_{ij;dc} &\sim_1 e_{ji;x}e_{ij;b}e_{ji;1}e_{ij;y}e_{ji;d}e_{ij;1} && \text{by Lemma 5.10(iii)}\\
&\sim_1 e_{ji;x}e_{ij;b}e_{ij;1}e_{ji;c}e_{ij;1} && \text{by (R1b)}_1\text{, noting that } c = yd\\
&\sim_1 e_{ji;x}e_{ji;1}e_{ij;b}e_{ji;c}e_{ij;1} && \text{by (R1a)}_1 \text{ twice}\\
&\sim_1 e_{ji;x}e_{ij;1}e_{ji;bc}e_{ij;1} && \text{by (R1b)}_1\\
&\sim_1 e_{ji;x}e_{ji;1}e_{ij;bc} && \text{by (R1b)}_1\\
&\sim_1 e_{ji;x}e_{ij;bc} && \text{by (R1a)}_1\\
&\sim_1 E_{ij;ac,bc} && \text{by Lemma 5.10(ii).}
\end{aligned}$$

In Case (ii), writing $a = xb$ and $d = yc$, we have

$$\begin{aligned}
E_{ij;ab}E_{ij;cd} &\sim_1 e_{ji;x}e_{ij;b}e_{ij;y}e_{ji;c}e_{ij;1} && \text{by Lemma 5.10(ii) and (iii)}\\
&\sim_1 e_{ji;x}e_{ji;1}e_{ij;b}e_{ji;c}e_{ij;1} && \text{by (R1a)}_1 \text{ twice}\\
&\sim_1 e_{ji;x}e_{ij;1}e_{ji;bc}e_{ij;1} && \text{by (R1b)}_1\\
&\sim_1 e_{ji;x}e_{ji;1}e_{ij;bc} && \text{by (R1b)}_1\\
&\sim_1 e_{ji;x}e_{ij;bc}, && \text{by (R1a)}_1\\
&\sim_1 E_{ij;ac,bc} && \text{by Lemma 5.10(ii),}
\end{aligned}$$

while

$$\begin{aligned}
E_{ji;ba}E_{ij;dc} &\sim_1 e_{ji;x}e_{ij;b}e_{ji;1}e_{ji;y}e_{ij;c} && \text{by Lemma 5.10(ii) and (iii)}\\
&\sim_1 e_{ji;x}e_{ji;1}e_{ij;b}e_{ji;1}e_{ij;c} && \text{by (R1a)}_1 \text{ twice}\\
&\sim_1 e_{ji;x}e_{ji;1}e_{ij;bc} && \text{by (R1b)}_1 \text{ twice}\\
&\sim_1 e_{ji;x}e_{ij;bc} && \text{by (R1a)}_1\\
&\sim_1 E_{ij;ac,bc} && \text{by Lemma 5.10(ii).}
\end{aligned}$$

In Case (iii), writing $b = xa$ and $c = yd$, we have

$$\begin{aligned}
E_{ij;ab}E_{ij;cd} &\sim_1 e_{ij;x}e_{ji;a}e_{ij;1}e_{ji;y}e_{ij;d} && \text{by Lemma 5.10(ii) and (iii)}\\
&\sim_1 e_{ij;x}e_{ij;1}e_{ji;a}e_{ij;1}e_{ji;y}e_{ij;d} && \text{by (R1a)}_1\\
&\sim_1 e_{ij;x}e_{ji;1}e_{ij;ayd} && \text{by (R1b)}_1 \text{ three times,}
\end{aligned}$$

while

$$\begin{aligned}
E_{ij;ac,bc} &\sim_1 e_{ij;x}e_{ji;ac}e_{ij;1} && \text{by Lemma 5.10(iii)}\\
&\sim_1 e_{ij;x}e_{ij;1}e_{ji;ac}e_{ij;1} && \text{by (R1a)}_1
\end{aligned}$$

$$\sim_1 e_{ij;x}e_{ji;1}e_{ij;ayd} \qquad \text{by (R1b)}_1 \text{ and } c = yd,$$

and

$$\begin{aligned} E_{ji;ba}E_{ij;dc} &\sim_1 e_{ij;x}e_{ji;a}e_{ij;y}e_{ji;d}e_{ij;1} && \text{by Lemma 5.10(ii) and (iii)} \\ &\sim_1 e_{ij;x}e_{ij;1}e_{ji;a}e_{ij;y}e_{ji;d}e_{ij;1} && \text{by (R1a)}_1 \\ &\sim_1 e_{ij;x}e_{ji;1}e_{ij;ayd} && \text{by (R1b)}_1 \text{ three times.} \end{aligned}$$

In Case (iv), writing $b = xa$ and $d = yc$, we have

$$\begin{aligned} E_{ij;ab}E_{ij;cd} &\sim_1 e_{ij;x}e_{ji;a}e_{ij;1}e_{ij;y}e_{ji;c}e_{ij;1} && \text{by Lemma 5.10(iii)} \\ &\sim_1 e_{ij;x}e_{ij;1}e_{ji;a}e_{ij;1}e_{ji;c}e_{ij;1} && \text{by (R1a)}_1 \text{ twice} \\ &\sim_1 e_{ij;x}e_{ji;1}e_{ij;ac} && \text{by (R1b)}_1 \text{ three times,} \end{aligned}$$

while

$$\begin{aligned} E_{ij;ac,bc} &\sim_1 e_{ij;x}e_{ji;ac}e_{ij;1} && \text{by Lemma 5.10(iii)} \\ &\sim_1 e_{ij;x}e_{ij;1}e_{ji;ac}e_{ij;1} && \text{by (R1a)}_1 \\ &\sim_1 e_{ij;x}e_{ji;1}e_{ij;ac} && \text{by (R1b)}_1, \end{aligned}$$

and

$$\begin{aligned} E_{ji;ba}E_{ij;dc} &\sim_1 e_{ij;x}e_{ji;a}e_{ji;y}e_{ij;c} && \text{by Lemma 5.10(ii)} \\ &\sim_1 e_{ij;x}e_{ij;1}e_{ji;a}e_{ij;c} && \text{by (R1a)}_1 \text{ twice} \\ &\sim_1 e_{ij;x}e_{ji;1}e_{ij;ac} && \text{by (R1b)}_1. \end{aligned}$$

(R2)$'_2$: This relation follows immediately from (R2)$_1$.

(R3a)$'_2$: If $a = xb$, then Lemma 5.10 and (R3a)$_1$ give $E_{ik;ab}E_{jk;1c} \sim_1 e_{ki;x}e_{ik;b}e_{jk;c} \sim_1 e_{ki;x}e_{ik;b} \sim_1 E_{ik;ab}$. An almost identical calculation deals with the case in which $b = xa$.

(R3b)$'_2$: If $a = xb$, then

$$\begin{aligned} E_{ki;ba}E_{ji;c1}E_{ik;11} &\sim_1 e_{ki;x}e_{ik;b}e_{ki;1}e_{ij;c}e_{ji;1}e_{ik;1} && \text{by Lemma 5.10(ii), (iii) and (iv)} \\ &\sim_1 e_{ki;x}e_{ik;b}e_{ik;1}e_{kj;c}e_{jk;1} && \text{by (R3b)}_1 \\ &\sim_1 e_{ki;x}e_{ik;b}e_{kj;c}e_{jk;1} && \text{by (R1a)}_1 \\ &\sim_1 E_{ik;ab}E_{jk;c1} && \text{by Lemma 5.10(ii).} \end{aligned}$$

If $b = xa$, then

$$\begin{aligned} E_{ik;ab}E_{jk;c1} &\sim_1 e_{ik;x}e_{ki;a}e_{ik;1}e_{kj;c}e_{jk;1} && \text{by Lemma 5.10(ii) and (iii)} \\ &\sim_1 e_{ik;x}e_{ki;a}e_{ki;1}e_{ij;c}e_{ji;1}e_{ik;1} && \text{by (R3b)}_1 \\ &\sim_1 e_{ik;x}e_{ki;a}e_{ij;c}e_{ji;1}e_{ik;1} && \text{by (R1a)}_1 \\ &\sim_1 E_{ki;ba}E_{ji;c1}E_{ik;11} && \text{by Lemma 5.10(ii) and (iv).} \end{aligned}$$

(R3c)$'_2$: Here we have

$$\begin{aligned} E_{ik;aa}E_{jk;b1} &\sim_1 e_{ki;1}e_{ik;a}e_{kj;b}e_{jk;1} && \text{by Lemma 5.10(ii)} \\ &\sim_1 e_{ik;1}e_{kj;b}e_{ji;a}e_{ij;1}e_{jk;1} && \text{by (R3c)}_1 \\ &\sim_1 e_{ik;1}e_{kj;b}e_{kj;1}e_{ji;a}e_{ij;1}e_{jk;1} && \text{by (R1a)}_1 \\ &\sim_1 e_{ik;1}e_{kj;b}e_{jk;1}e_{ki;a}e_{ik;1}e_{kj;1}e_{jk;1} && \text{by (R3b)}_1 \\ &\sim_1 e_{ik;1}e_{kj;b}e_{jk;1}e_{ki;a}e_{ik;1}e_{jk;1} && \text{by (R1e)}_1 \\ &\sim_1 e_{ik;1}e_{kj;b}e_{jk;1}e_{ki;a}e_{ik;1} && \text{by (R3a)}_1 \\ &\sim_1 E_{ik;11}E_{jk;b1}E_{ik;a1} && \text{by Lemma 5.10(ii) and (iv).} \end{aligned}$$

(R4a)$'_2$: As explained in Remark 5.3, we only need to show that $E_{ij;ab}E_{ik;cd} \sim_1 E_{ik;ac,d}E_{ij;1,bc}$. However, to do this, we must consider six cases:

(i) $a \leq_{\mathscr{L}} b$ and $c \leq_{\mathscr{L}} d$; (iii) $a \leq_{\mathscr{L}} b$ and $d \leq_{\mathscr{L}} ac$; (v) $a \leq_{\mathscr{L}} b$ and $ac \leq_{\mathscr{L}} d \leq_{\mathscr{L}} c$;

(ii) $b \leq_{\mathscr{L}} a$ and $c \leq_{\mathscr{L}} d$; (iv) $b \leq_{\mathscr{L}} a$ and $d \leq_{\mathscr{L}} ac$; (vi) $b \leq_{\mathscr{L}} a$ and $ac \leq_{\mathscr{L}} d \leq_{\mathscr{L}} c$.

(Note that $ac \leq_{\mathscr{L}} d$ in Cases (i) and (ii), and that $d \leq_{\mathscr{L}} c$ in Cases (iii) and (iv).) For Cases (i) and (ii), write $c = yd$, and note that $ac = (ay)d$, so that

$$\begin{aligned}
E_{ik;ac,d}E_{ij;1,bc} &\sim e_{ki;ay}e_{ik;d}e_{ij;bc} && \text{by Lemma 5.10(ii) and (iv)}\\
&= e_{ki;ay}e_{ik;d}e_{ij;byd} && \\
&\sim e_{ki;ay}e_{kj;by}e_{ik;d} && \text{by (R4)}_1.
\end{aligned}$$

In Case (i), writing $a = xb$, we have

$$\begin{aligned}
E_{ij;ab}E_{ik;cd} &\sim_1 e_{ji;x}e_{ij;b}e_{ki;y}e_{ik;d} && \text{by Lemma 5.10(ii)}\\
&\sim_1 e_{ji;x}e_{ki;y}e_{kj;by}e_{ik;d} && \text{by (R4)}_1\\
&\sim_1 e_{ji;x}e_{kj;by}e_{ik;d} && \text{by (R3a)}_1\\
&\sim_1 e_{ki;ay}e_{kj;by}e_{ik;d} && \text{by (R4)}_1\text{, noting that } xby = ay.
\end{aligned}$$

In Case (ii), writing $b = xa$, we have

$$\begin{aligned}
E_{ij;ab}E_{ik;cd} &\sim_1 e_{ij;x}e_{ji;a}e_{ij;1}e_{ki;y}e_{ik;d} && \text{by Lemma 5.10(ii) and (iii)}\\
&\sim_1 e_{ij;x}e_{ji;a}e_{ki;y}e_{kj;y}e_{ik;d} && \text{by (R4)}_1\\
&\sim_1 e_{ij;x}e_{ji;a}e_{kj;y}e_{ik;d} && \text{by (R3a)}_1\\
&\sim_1 e_{ij;x}e_{ki;ay}e_{kj;y}e_{ik;d} && \text{by (R4)}_1\\
&\sim_1 e_{ki;ay}e_{kj;by}e_{kj;y}e_{ik;d} && \text{by (R4)}_1\text{, noting that } xay = by\\
&\sim_1 e_{ki;ay}e_{kj;by}e_{ik;d} && \text{by (R1a)}_1.
\end{aligned}$$

For Case (iii), write $a = xb$ and $d = zac$. Then

$$\begin{aligned}
E_{ij;ab}E_{ik;cd} &\sim_1 e_{ji;x}e_{ij;b}e_{ik;za}e_{ki;c}e_{ik;1} && \text{by Lemma 5.10(ii) and (iii)}\\
&= e_{ji;x}e_{ij;b}e_{ik;zxb}e_{ki;c}e_{ik;1} && \\
&\sim_1 e_{ji;x}e_{jk;zx}e_{ij;b}e_{ki;c}e_{ik;1} && \text{by (R4)}_1\\
&\sim_1 e_{jk;zx}e_{ji;x}e_{ki;c}e_{kj;bc}e_{ik;1} && \text{by (R4)}_1 \text{ twice}\\
&\sim_1 e_{jk;zx}e_{ji;x}e_{kj;bc}e_{ik;1} && \text{by (R3a)}_1\\
&\sim_1 e_{ji;x}e_{jk;zx}e_{ik;1}e_{ij;bc} && \text{by (R4)}_1 \text{ twice,}
\end{aligned}$$

and

$$\begin{aligned}
E_{ik;ac,d}E_{ij;1,bc} &\sim e_{ik;z}e_{ki;ac}e_{ik;1}e_{ij;bc} && \text{by Lemma 5.10(iii) and (iv)}\\
&\sim e_{ik;z}e_{ki;xbc}e_{kj;bc}e_{ik;1} && \text{by (R4)}_1 \text{ and } a = xb\\
&\sim e_{ik;z}e_{ji;x}e_{kj;bc}e_{ik;1} && \text{by (R4)}_1\\
&\sim e_{ji;x}e_{jk;zx}e_{ik;1}e_{ij;bc} && \text{by (R4)}_1 \text{ twice.}
\end{aligned}$$

For Case (iv), write $b = xa$ and $d = zac$. Then

$$\begin{aligned}
E_{ij;ab}E_{ik;cd} &\sim_1 e_{ij;x}e_{ji;a}e_{ij;1}e_{ik;za}e_{ki;c}e_{ik;1} && \text{by Lemma 5.10(iii)}\\
&\sim_1 e_{ij;x}e_{ji;a}e_{jk;za}e_{ij;1}e_{ki;c}e_{ik;1} && \text{by (R4)}_1\\
&\sim_1 e_{ij;x}e_{jk;za}e_{ji;a}e_{ki;c}e_{kj;c}e_{ik;1} && \text{by (R4)}_1 \text{ twice}\\
&\sim_1 e_{ij;x}e_{jk;za}e_{ji;a}e_{kj;c}e_{ik;1} && \text{by (R3a)}_1\\
&\sim_1 e_{ij;x}e_{jk;za}e_{ki;ac}e_{kj;c}e_{ik;1} && \text{by (R4)}_1\\
&\sim_1 e_{ij;x}e_{kj;1}e_{jk;za}e_{kj;c}e_{ki;ac}e_{ik;1} && \text{by (R3a)}_1 \text{ and (R4)}_1
\end{aligned}$$

$$
\begin{aligned}
&\sim_1 e_{ij;x}e_{jk;1}e_{kj;zac}e_{ki;ac}e_{ik;1} && \text{by (R1b)}_1\\
&\sim_1 e_{ij;x}e_{kj;1}e_{ik;z}e_{ki;ac}e_{ik;1} && \text{by Lemma 5.11(iii)}\\
&\sim_1 e_{ij;x}e_{ik;z}e_{ki;ac}e_{ik;1} && \text{by (R3a)}_1,
\end{aligned}
$$

and

$$
\begin{aligned}
E_{ik;ac,d}E_{ij;1,bc} &\sim e_{ik;z}e_{ki;ac}e_{ik;1}e_{ij;bc} && \text{by Lemma 5.10(iii) and (iv)}\\
&\sim e_{ik;z}e_{ki;ac}e_{kj;xac}e_{ik;1} && \text{by (R4)}_1 \text{ and } b = xa\\
&\sim e_{ik;z}e_{ij;x}e_{ki;ac}e_{ik;1} && \text{by (R4)}_1\\
&\sim e_{ij;x}e_{ik;z}e_{ki;ac}e_{ik;1} && \text{by (R4)}_1.
\end{aligned}
$$

For Case (v), write $a = xb$, $d = yc$ and $ac = zd$. Then

$$
\begin{aligned}
E_{ij;ab}E_{ik;cd} &\sim_1 e_{ji;x}e_{ij;b}e_{ik;y}e_{ki;c}e_{ik;1} && \text{by Lemma 5.10(ii) and (iii)}\\
&\sim_1 e_{ji;x}e_{ik;y}e_{ij;b}e_{ki;c}e_{ik;1} && \text{by (R4)}_1\\
&\sim_1 e_{ji;x}e_{ik;y}e_{ki;c}e_{kj;bc}e_{ik;1} && \text{by (R4)}_1\\
&\sim_1 e_{ji;x}e_{ki;1}e_{ik;y}e_{ki;c}e_{ik;1}e_{ij;bc} && \text{by (R3a)}_1 \text{ and (R4)}_1\\
&\sim_1 e_{ji;x}e_{ik;1}e_{ki;d}e_{ik;1}e_{ij;bc} && \text{by (R1b)}_1 \text{ and } d = yc\\
&\sim_1 e_{ji;x}e_{ki;1}e_{ik;d}e_{ij;bc} && \text{by (R1b)}_1\\
&\sim_1 e_{ji;x}e_{ik;d}e_{ij;bc} && \text{by (R3a)}_1\\
&\sim_1 e_{ki;z}e_{ik;d}e_{ij;bc} && \text{by Lemma 5.11(iv), as } zd = ac = xbc\\
&\sim_1 E_{ik;ac,d}E_{ij;1,bc} && \text{by Lemma 5.10(ii) and (iv).}
\end{aligned}
$$

For Case (vi), write $b = xa$, $d = yc$ and $ac = zd$. Then

$$
\begin{aligned}
E_{ij;ab}E_{ik;cd} &\sim_1 e_{ij;x}e_{ji;a}e_{ij;1}e_{ik;y}e_{ki;c}e_{ik;1} && \text{by Lemma 5.10(iii)}\\
&\sim_1 e_{ij;x}e_{ji;a}e_{jk;y}e_{ij;1}e_{ki;c}e_{ik;1} && \text{by (R4)}_1\\
&\sim_1 e_{ij;x}e_{jk;y}e_{ji;a}e_{ki;c}e_{kj;c}e_{ik;1} && \text{by (R4)}_1 \text{ twice}\\
&\sim_1 e_{ij;x}e_{jk;y}e_{ji;a}e_{kj;c}e_{ik;1} && \text{by (R3a)}_1\\
&\sim_1 e_{ij;x}e_{kj;1}e_{jk;y}e_{kj;c}e_{ki;ac}e_{ik;1} && \text{by (R3a)}_1 \text{ and (R4)}_1\\
&\sim_1 e_{ij;x}e_{jk;1}e_{kj;d}e_{ki;zd}e_{ik;1} && \text{by (R1b)}_1,\ yc = d \text{ and } ac = zd\\
&\sim_1 e_{ij;x}e_{jk;1}e_{ji;z}e_{kj;d}e_{ik;1} && \text{by (R4)}_1\\
&\sim_1 e_{ij;x}e_{ki;z}e_{jk;1}e_{kj;d}e_{ik;1} && \text{by (R4)}_1\\
&\sim_1 e_{ki;z}e_{kj;xz}e_{jk;1}e_{ik;1}e_{ij;d} && \text{by (R4)}_1 \text{ twice}\\
&\sim_1 e_{ki;z}e_{kj;xz}e_{jk;1}e_{ij;d} && \text{by (R3a)}_1\\
&\sim_1 e_{ki;z}e_{kj;xz}e_{ij;d}e_{ik;d} && \text{by (R4)}_1\\
&\sim_1 e_{ki;z}e_{kj;xz}e_{ik;d} && \text{by (R3a)}_1\\
&\sim_1 e_{ki;z}e_{ik;d}e_{ij;bc} && \text{by (R4)}_1 \text{ and } xzd = xac = bc\\
&\sim_1 E_{ik;ac,d}E_{ij;1,bc} && \text{by Lemma 5.10(ii) and (iv).}
\end{aligned}
$$

(R4b)$'_2$: As in Remark 5.3, we only need to show that $E_{ik;c,bd}E_{ij;1,ad} \sim_1 E_{jk;ab}E_{ij;cd}$. Again, we must consider six cases:

(i) $a \leq_{\mathscr{L}} b$ and $d \leq_{\mathscr{L}} c$;
(ii) $b \leq_{\mathscr{L}} a$ and $d \leq_{\mathscr{L}} c$;
(iii) $a \leq_{\mathscr{L}} b$ and $c \leq_{\mathscr{L}} bd$;
(iv) $b \leq_{\mathscr{L}} a$ and $c \leq_{\mathscr{L}} bd$;
(v) $a \leq_{\mathscr{L}} b$ and $bd \leq_{\mathscr{L}} c \leq_{\mathscr{L}} d$;
(vi) $b \leq_{\mathscr{L}} a$ and $bd \leq_{\mathscr{L}} c \leq_{\mathscr{L}} d$.

(Note that $bd \leq_{\mathscr{L}} c$ in Cases (i) and (ii), and that $c \leq_{\mathscr{L}} d$ in Cases (iii) and (iv).) For Cases (i) and (ii), write $d = yc$, and note that $bd = (by)c$, so that

$$
E_{ik;c,bd}E_{ij;1,ad} \sim_1 e_{ik;by}e_{ki;c}e_{ik;1}e_{ij;ad} \qquad \text{by Lemma 5.10(iii) and (iv)}
$$

$$\begin{aligned}
&\sim_1 e_{ik;by}e_{ik;1}e_{ki;c}e_{ik;1}e_{ij;ad} && \text{by (R1a)}_1\\
&\sim_1 e_{ik;by}e_{ki;1}e_{ik;c}e_{ij;ayc} && \text{by (R1b)}_1 \text{ and } d = yc\\
&\sim_1 e_{ik;by}e_{ik;1}e_{ij;ay}e_{ji;c}e_{ij;1} && \text{by Lemma 5.11(ii)}\\
&\sim_1 e_{ik;by}e_{ij;ay}e_{ji;c}e_{ij;1} && \text{by (R1a)}_1.
\end{aligned}$$

In Case (i), writing $a = xb$, we have

$$\begin{aligned}
E_{jk;ab}E_{ij;cd} &\sim_1 e_{kj;x}e_{jk;b}e_{ij;y}e_{ji;c}e_{ij;1} && \text{by Lemma 5.10(ii) and (iii)}\\
&\sim_1 e_{kj;x}e_{ij;y}e_{ik;by}e_{ji;c}e_{ij;1} && \text{by (R4)}_1\\
&\sim_1 e_{kj;x}e_{ik;by}e_{ji;c}e_{ij;1} && \text{by (R3a)}_1\\
&\sim_1 e_{ik;by}e_{ij;ay}e_{ji;c}e_{ij;1} && \text{by (R4)}_1 \text{ and } xb = a.
\end{aligned}$$

In Case (ii), writing $b = xa$, we have

$$\begin{aligned}
E_{jk;ab}E_{ij;cd} &\sim_1 e_{jk;x}e_{kj;a}e_{jk;1}e_{ij;y}e_{ji;c}e_{ij;1} && \text{by Lemma 5.10(iii)}\\
&\sim_1 e_{jk;x}e_{kj;a}e_{ij;y}e_{ik;y}e_{ji;c}e_{ij;1} && \text{by (R4)}_1\\
&\sim_1 e_{jk;x}e_{kj;a}e_{ik;y}e_{ji;c}e_{ij;1} && \text{by (R3a)}_1\\
&\sim_1 e_{jk;x}e_{ik;y}e_{ij;ay}e_{ji;c}e_{ij;1} && \text{by (R4)}_1\\
&\sim_1 e_{jk;x}e_{ij;ay}e_{ji;c}e_{ij;1} && \text{by (R3a)}_1\\
&\sim_1 e_{ik;by}e_{ij;ay}e_{ji;c}e_{ij;1} && \text{by (R4)}_1 \text{ and } xa = b.
\end{aligned}$$

For Case (iii), write $a = xb$ and $c = zbd$. Then

$$\begin{aligned}
E_{jk;ab}E_{ij;cd} &\sim_1 e_{kj;x}e_{jk;b}e_{ji;zb}e_{ij;d} && \text{by Lemma 5.10(ii)}\\
&\sim_1 e_{kj;x}e_{ji;zb}e_{jk;b}e_{ij;d} && \text{by (R4)}_1\\
&\sim_1 e_{kj;x}e_{ij;1}e_{ji;zb}e_{ij;d}e_{ik;bd} && \text{by (R3a)}_1 \text{ and (R4)}_1\\
&\sim_1 e_{kj;x}e_{ji;1}e_{ij;zbd}e_{ik;bd} && \text{by (R1b)}_1\\
&\sim_1 e_{kj;x}e_{ji;1}e_{kj;z}e_{ik;bd} && \text{by (R4)}_1\\
&\sim_1 e_{kj;x}e_{kj;z}e_{ki;z}e_{ik;bd} && \text{by (R4)}_1\\
&\sim_1 e_{kj;x}e_{ki;z}e_{ik;bd} && \text{by (R1a)}_1\\
&\sim_1 e_{ki;z}e_{kj;x}e_{ik;bd} && \text{by (R4)}_1\\
&\sim_1 e_{ki;z}e_{ik;bd}e_{ij;ad} && \text{by (R4)}_1 \text{ and } xb = a\\
&\sim_1 E_{ik;c,bd}E_{ij;1,ad} && \text{by Lemma 5.10(ii) and (iv).}
\end{aligned}$$

For Case (iv), write $b = xa$ and $c = zbd$. Then

$$\begin{aligned}
E_{jk;ab}E_{ij;cd} &\sim_1 e_{jk;x}e_{kj;a}e_{jk;1}e_{ji;zb}e_{ij;d} && \text{by Lemma 5.10(ii) and (iii)}\\
&\sim_1 e_{jk;x}e_{kj;a}e_{ji;zb}e_{jk;1}e_{ij;d} && \text{by (R4)}_1\\
&\sim_1 e_{jk;x}e_{kj;a}e_{ij;1}e_{ji;zb}e_{ij;d}e_{ik;d} && \text{by (R3a)}_1 \text{ and (R4)}_1\\
&\sim_1 e_{jk;x}e_{kj;a}e_{ji;1}e_{ij;zbd}e_{ik;d} && \text{by (R1b)}_1\\
&\sim_1 e_{jk;x}e_{kj;a}e_{ji;1}e_{kj;zb}e_{ik;d} && \text{by (R4)}_1\\
&\sim_1 e_{jk;x}e_{kj;a}e_{kj;zb}e_{ki;zb}e_{ik;d} && \text{by (R4)}_1\\
&\sim_1 e_{jk;x}e_{kj;a}e_{ki;zxa}e_{ik;d} && \text{by (R1a)}_1 \text{ and } b = xa\\
&\sim_1 e_{jk;x}e_{ji;zx}e_{kj;a}e_{ik;d} && \text{by (R4)}_1\\
&\sim_1 e_{ji;zx}e_{jk;x}e_{ik;d}e_{ij;ad} && \text{by (R4)}_1 \text{ twice}\\
&\sim_1 e_{ji;zx}e_{jk;x}e_{ij;ad} && \text{by (R3a)}_1\\
&\sim_1 e_{ki;z}e_{jk;x}e_{ij;ad} && \text{by (R4)}_1\\
&\sim_1 e_{ki;z}e_{ik;bd}e_{ij;ad} && \text{by (R4)}_1 \text{ and } xa = b\\
&\sim_1 E_{ik;c,bd}E_{ij;1,ad} && \text{by Lemma 5.10(ii) and (iv).}
\end{aligned}$$

For Case (v), write $a = xb$, $c = yd$ and $bd = zc$. Then

$$\begin{aligned}
E_{jk;ab}E_{ij;cd} &\sim_1 e_{kj;x}e_{jk;b}e_{ji;y}e_{ij;d} && \text{by Lemma 5.10(ii)}\\
&\sim_1 e_{kj;x}e_{ji;y}e_{jk;b}e_{ij;d} && \text{by (R4)}_1\\
&\sim_1 e_{kj;x}e_{ij;1}e_{ji;y}e_{ij;d}e_{ik;bd} && \text{by (R3a)}_1 \text{ and (R4)}_1\\
&\sim_1 e_{kj;x}e_{ji;1}e_{ij;c}e_{ik;zc} && \text{by (R1b)}_1,\ yd = c \text{ and } bd = zc,
\end{aligned}$$

and

$$\begin{aligned}
E_{ik;c,bd}E_{ij;1,ad} &\sim_1 e_{ik;z}e_{ki;c}e_{ik;1}e_{ij;ad} && \text{by Lemma 5.10(iii) and (iv)}\\
&\sim_1 e_{ik;z}e_{ki;c}e_{kj;xzc}e_{ik;1} && \text{by (R4)}_1 \text{ and } ad = xbd = xzc\\
&\sim_1 e_{ik;z}e_{ij;xz}e_{ki;c}e_{ik;1} && \text{by (R4)}_1\\
&\sim_1 e_{kj;x}e_{ik;z}e_{ki;c}e_{ik;1} && \text{by (R4)}_1\\
&\sim_1 e_{kj;x}e_{ij;1}e_{ik;z}e_{ki;c}e_{ik;1} && \text{by (R3a)}_1\\
&\sim_1 e_{kj;x}e_{ji;1}e_{ij;c}e_{ik;zc} && \text{by Lemma 5.11(ii).}
\end{aligned}$$

For Case (vi), write $b = xa$, $c = yd$ and $bd = zc$. Then

$$\begin{aligned}
E_{jk;ab}E_{ij;cd} &\sim_1 e_{jk;x}e_{kj;a}e_{jk;1}e_{ji;y}e_{ij;d} && \text{by Lemma 5.10(ii) and (iii)}\\
&\sim_1 e_{jk;x}e_{kj;a}e_{ki;y}e_{jk;1}e_{ij;d} && \text{by (R4)}_1\\
&\sim_1 e_{jk;x}e_{ki;y}e_{kj;a}e_{ij;d}e_{ik;d} && \text{by (R4)}_1 \text{ twice}\\
&\sim_1 e_{jk;x}e_{ki;y}e_{kj;a}e_{ik;d} && \text{by (R3a)}_1\\
&\sim_1 e_{jk;x}e_{kj;a}e_{ki;y}e_{ik;d} && \text{by (R4)}_1\\
&\sim_1 e_{jk;x}e_{kj;ad}e_{ki;c}e_{ik;1} && \text{by Lemma 5.11(i) and } yd = c\\
&\sim_1 e_{jk;x}e_{ki;c}e_{kj;ad}e_{ik;1} && \text{by (R4)}_1\\
&\sim_1 e_{ik;z}e_{ki;c}e_{kj;ad}e_{ik;1} && \text{by Lemma 5.11(iv), as } zc = bd = xad\\
&\sim_1 e_{ik;z}e_{ki;c}e_{ik;1}e_{ij;ad} && \text{by (R4)}_1\\
&\sim_1 E_{ik;c,bd}E_{ij;1,ad} && \text{by Lemma 5.10(iii) and (iv).}
\end{aligned}$$

This completes the proof of the theorem. □

As noted at the beginning of this section, some of the relations from R_1 may be simplified in the case that M is a group.

Theorem 5.12. *If M is a group, then $M \wr \mathrm{Sing}_n$ has presentation $\langle X_1 : R_1' \rangle$ via ϕ_1, where R_1' is obtained from R_1 by replacing* (R1a–R1e)$_1$ *by*

$$e_{ij;a}e_{ij;b} = e_{ij;a} = e_{ji;a^{-1}}e_{ij;a} \qquad \textit{for } a, b \in M \textit{ and distinct } i, j \qquad \text{(R1a)}_1'$$

$$e_{ij;1}e_{ji;a}e_{ij;b} = e_{ji;1}e_{ij;ab} \qquad \textit{for } a, b \in M \textit{ and distinct } i, j. \qquad \text{(R1b)}_1$$

Proof. Suppose M is a group. We start with the presentation $\langle X_1 : R_1 \rangle$ from Therorem 5.9. Since $(e_{ji;a^{-1}}e_{ij;a})\phi_1 = e_{ij;a}\phi_1$, we may augment (R1a)$_1$ by replacing it with (R1a)$_1'$. Relation (R1e)$_1$ may be removed, as it is just part of (R1a)$_1'$. Relation (R1c)$_1$ may also be removed, since $ac = bc$ is only possible in M (a group) if $a = b$, in which case the relation is vacuous. For relation (R1d)$_1$, let $i, j \in \mathbf{n}$ be distinct, and let $a, b, c \in M$ with $abc = c$, noting that this forces $b = a^{-1}$. Then, writing $\sim_1' = (R_1')^\sharp$, we have

$$\begin{aligned}
e_{ji;a}e_{ij;bc} &\sim_1' e_{ij;b}e_{ji;a}e_{ij;bc} && \text{by (R1a)}_1' \text{ and } b = a^{-1}\\
&\sim_1' e_{ij;b}e_{ij;1}e_{ji;a}e_{ij;bc} && \text{by (R1a)}_1'\\
&\sim_1' e_{ij;b}e_{ji;1}e_{ij;c} && \text{by (R1b)}_1 \text{ and } abc = c\\
&\sim_1' e_{ij;b}e_{ij;1}e_{ji;c}e_{ij;1} && \text{by (R1b)}_1\\
&\sim_1' e_{ij;b}e_{ji;c}e_{ij;1} && \text{by (R1a)}_1'.
\end{aligned}$$

□

Remark 5.13. Theorem 5.12 can also be deduced directly from Theorem 5.2 in a similar way to Theorem 5.9, though the calculations are far easier under the assumption that M is a group. Here we may simply define the words $E_{ij;ab}$ as $e_{ji;ab^{-1}}e_{ij;b}$ for all i, j, a, b, and we never need to consider multiple cases (according to whether $a \leq_{\mathscr{L}} b$ or $b \leq_{\mathscr{L}} a$, etc.).

Remark 5.14. Although $M \wr \mathrm{Sing}_n \neq \langle \mathcal{X}_1 \rangle$ in the case that $M/\mathscr{L}$ is not a chain, it is still the case (by Theorem 4.7) that $\langle E(M \wr \mathrm{Sing}_n) \rangle = \langle \mathcal{X}_1 \rangle$. It would be interesting to give a presentation for $\langle E(M \wr \mathrm{Sing}_n) \rangle$ in terms of the generating set $\mathcal{X}_1$.

6 The endomorphism monoid of a uniform partition

We now apply the results of previous sections to obtain a presentation for the idempotent generated subsemigroup of the endomorphism monoid of a uniform partition of a finite set. These monoids are defined as follows. Let X be a non-empty finite set, and let $\mathcal{P} = \{C_1, \ldots, C_n\}$ be a uniform partition of X: by this we mean that the sets $C_1, \ldots, C_n$ are pairwise disjoint, have a common size (m, say), and their union is all of X (so that $|X| = mn$). The *endomorphism monoid of* $\mathcal{P}$ is defined to be the submonoid

$$\mathcal{T}(X, \mathcal{P}) = \{\alpha \in \mathcal{T}_X : (\forall i \in \mathbf{n})(\exists j \in \mathbf{n})\ C_i\alpha \subseteq C_j\}$$

of $\mathcal{T}_X$, the full transformation semigroup on X. These monoids were introduced by Pei in [68], and have subsequently been studied by a number of different authors. In particular, the rank of $\mathcal{T}(X, \mathcal{P})$ was calculated in [5], while the rank and idempotent rank of the idempotent generated subsemigroup of $\mathcal{T}(X, \mathcal{P})$ were calculated in [17]; for the corresponding studies of the non-uniform case, see [4, 18]. As noted in [5], $\mathcal{T}(X, \mathcal{P})$ is isomorphic to the wreath product $\mathcal{T}_m \wr \mathcal{T}_n$. So we will concentrate on such a wreath product $\mathcal{T}_m \wr \mathcal{T}_n$, and our goal (as stated above) is to give a monoid presentation for the idempotent generated subsemigroup $\langle E(\mathcal{T}_m \wr \mathcal{T}_n) \rangle$. In fact, we are able to solve a more general problem: namely, in Theorem 6.3, we give a monoid presentation for the idempotent generated subsemigroup $\langle E(M \wr \mathcal{T}_n) \rangle$ of $M \wr \mathcal{T}_n$, modulo a presentation for $\langle E(M) \rangle$, in the case that M is a monoid satisfying $\langle E(M) \rangle = \{1\} \cup (M \setminus G)$, where G is the group of units of M. So for the remainder of this section, we fix a monoid M, write G for its group of units, and we assume that $\langle E(M) \rangle = \{1\} \cup (M \setminus G)$. Examples of such monoids M include the finite full transformation semigroups [49], finite dimensional full linear monoids [31], finite partition monoids [25], finite Brauer monoids [61], the endomorphism monoids of certain finite dimensional free M-acts (see Corollary 4.9 above), and many more. Presentations for the idempotent generated subsemigroups of some (but not all) of these examples are known [25–27, 61].

To avoid confusion, we will write 1 and 1_n for the identity elements of M and $\mathcal{T}_n$, respectively. Recall that any subsemigroup S of $\mathcal{T}_n$ leads to a wreath product $K \wr S$, for any monoid K. In particular, when $S = \{1_n\} \subseteq \mathcal{T}_n$ consists of only the identity transformation, $K \wr \{1_n\}$ is isomorphic to the direct product of n copies of K. The $M = \mathcal{T}_m$ case of the next result is contained in [17, Proposition 4.1]. We write $A = B \sqcup C$ to indicate that A is the disjoint union of B and C. To simplify notation throughout this section, if T is any semigroup, we will write $\mathbb{E}(T) = \langle E(T) \rangle$ for the idempotent generated subsemigroup of T.

Proposition 6.1. *Suppose M is a monoid with group of units G, and that $\mathbb{E}(M) = \{1\} \cup (M \setminus G)$. Then*

(i) $\mathbb{E}(M \wr \mathcal{T}_n) = (\mathbb{E}(M) \wr \{1_n\}) \sqcup (M \wr \mathrm{Sing}_n)$; *and*

(ii) $M \wr \mathrm{Sing}_n = (\mathbb{E}(M) \wr \{1_n\})(G \wr \mathrm{Sing}_n)$.

Proof. We begin with (ii). First note that if $(\mathbf{a}, 1_n) \in \mathbb{E}(M) \wr \{1_n\}$ and $(\mathbf{b}, \beta) \in G \wr \mathrm{Sing}_n$, then $(\mathbf{a}, 1_n)(\mathbf{b}, \beta) = (\mathbf{ab}, \beta) \in M \wr \mathrm{Sing}_n$. Conversely, suppose $(\mathbf{c}, \gamma) \in M \wr \mathrm{Sing}_n$. For each $i \in \mathbf{n}$, define

$$a_i = \begin{cases} 1 & \text{if } c_i \in G \\ c_i & \text{if } c_i \in M \setminus G \end{cases} \qquad \text{and} \qquad b_i = \begin{cases} c_i & \text{if } c_i \in G \\ 1 & \text{if } c_i \in M \setminus G. \end{cases}$$

Since $a_ib_i = c_i$ for each i, it follows that $(\mathbf{c}, \gamma) = (\mathbf{a}, 1_n)(\mathbf{b}, \gamma)$. It is also clear that $(\mathbf{b}, \gamma) \in G \wr \mathrm{Sing}_n$, while $(\mathbf{a}, 1_n) \in \mathbb{E}(M) \wr \{1_n\}$ follows from the fact that $\mathbb{E}(M) = \{1\} \cup (M \setminus G)$. This completes the proof of (ii).

For (i), first suppose $(\mathbf{a}_1,\alpha_1),\dots,(\mathbf{a}_k,\alpha_k) \in E(M \wr \mathcal{T}_n)$, and write $(\mathbf{a},\alpha) = (\mathbf{a}_1,\alpha_1)\cdots(\mathbf{a}_k,\alpha_k)$. If any of $\alpha_1,\dots,\alpha_k$ belongs to Sing_n, then so too does $\alpha = \alpha_1\cdots\alpha_k$, so that $(\mathbf{a},\alpha) \in M \wr \mathrm{Sing}_n$. On the other hand, if $\alpha_1 = \cdots = \alpha_k = 1_n$, then Lemma 4.1 gives $\mathbf{a}_i \in E(M)^n$ for all i, in which case

$$(\mathbf{a},\alpha) = (\mathbf{a}_1\cdots\mathbf{a}_k, 1_n) \in \mathbb{E}(M) \wr \{1_n\}.$$

We have shown that $\mathbb{E}(M \wr \mathcal{T}_n) \subseteq (\mathbb{E}(M) \wr \{1_n\}) \sqcup (M \wr \mathrm{Sing}_n)$. To prove the converse, first suppose $(\mathbf{b}, 1_n) \in \mathbb{E}(M) \wr \{1_n\}$. Since $1 \in E(M)$, we may write $\mathbf{b} = \mathbf{b}_1\cdots\mathbf{b}_k$ with $\mathbf{b}_1,\dots,\mathbf{b}_k \in E(M)^n$, and it then follows that $(\mathbf{b},1_n) = (\mathbf{b}_1,1_n)\cdots(\mathbf{b}_k,1_n) \in \mathbb{E}(M \wr \mathcal{T}_n)$. This shows that $\mathbb{E}(M) \wr \{1_n\} \subseteq \mathbb{E}(M \wr \mathcal{T}_n)$. Together with Part (ii) and the fact that $G \wr \mathrm{Sing}_n$ is idempotent generated (by Theorem 4.7(iii)), it also follows that $M \wr \mathrm{Sing}_n \subseteq \mathbb{E}(M \wr \mathcal{T}_n)$. $\square$

Suppose now that $\mathbb{E}(M) = \langle E(M)\rangle$ has monoid presentation $\langle Y : Q\rangle$ via $\psi : Y^* \to \mathbb{E}(M)$. Since this is a *monoid* presentation, we may assume that $y\psi \neq 1$ for all $y \in Y$, and it will be important to do so in what follows. Define new alphabets $Y_{(i)} = \{y_{(i)} : y \in Y\}$ for each $i \in \mathbf{n}$, and put $\mathbf{Y} = Y_{(1)} \cup \cdots \cup Y_{(n)}$. For a word $w = y_1\cdots y_k \in Y^*$, and for $i \in \mathbf{n}$, define $w_{(i)} = (y_1)_{(i)}\cdots(y_k)_{(i)} \in Y_{(i)}^*$. For each $i \in \mathbf{n}$, write $Q_{(i)} = \{(u_{(i)}, v_{(i)}) : (u,v) \in Q\}$ and put $\mathbf{Q} = Q_{(1)} \cup \cdots \cup Q_{(n)}$. We also define

$$R_{\leftrightarrow} = \{(x_{(i)}y_{(j)}, y_{(j)}x_{(i)}) : x,y \in Y,\ i,j \in \mathbf{n},\ i \neq j\}.$$

For $a \in M$ and $i \in \mathbf{n}$, write $a_{(i)} = ((1,\dots,1,a,1,\dots,1),1_n)$, where the a is in the ith position. Define an epimorphism

$$\Psi : \mathbf{Y}^* \to \mathbb{E}(M) \wr \{1_n\} : y_{(i)} \mapsto (y\psi)_{(i)}.$$

The next result follows from an obvious (and essentially folklore) result on presentations for direct products of monoids.

Lemma 6.2. *With the above notation, the monoid $\mathbb{E}(M) \wr \{1_n\}$ has monoid presentation $\langle \mathbf{Y} : \mathbf{Q} \cup R_{\leftrightarrow}\rangle$ via Ψ.* $\square$

We fix the semigroup presentation $\langle X_1 : R_1'\rangle$ for $G \wr \mathrm{Sing}_n$ (via $\phi_1 : X_1^+ \to G \wr \mathrm{Sing}_n$) from Theorem 5.12, where $X_1 = \{e_{ij;a} : i,j \in \mathbf{n},\ i \neq j,\ a \in G\}$, and so on. We now explain how to stitch this together with the monoid presentation $\langle \mathbf{Y} : \mathbf{Q} \cup R_{\leftrightarrow}\rangle$ for $\mathbb{E}(M) \wr \{1_n\}$ in order to yield a monoid presentation for $\mathbb{E}(M \wr \mathcal{T}_n)$. In what follows, it will be convenient to write $\overline{w} = w\psi \in \mathbb{E}(M)$ for any word $w \in Y^*$. By Proposition 6.1, we may define an epimorphism

$$\Theta : (\mathbf{Y} \cup X_1)^* \to \mathbb{E}(M \wr \mathcal{T}_n) : y_{(i)} \mapsto \overline{y}_{(i)},\ e_{ij;a} \mapsto \varepsilon_{ij;a}.$$

We will also choose (and fix for the remainder of the section) a set of words $\{h_a : a \in \mathbb{E}(M)\} \subseteq Y^*$ such that $h_a\psi = a$ for all $a \in \mathbb{E}(M)$. For $a \in \mathbb{E}(M)$ and $i \in \mathbf{n}$, define $h_{a;i} = (h_a)_{(i)} \in Y_{(i)}^*$, noting that $h_{a;i}\Theta = h_{a;i}\Psi = a_{(i)}$. In practice, we might like to choose the words h_a to be as short as possible, but this is not a requirement. Note that if $w \in \mathbf{Y}^*$ is such that $w\Theta = ((a_1,\dots,a_n),1_n) \in \mathbb{E}(M) \wr \{1_n\}$, then w may be transformed into $h_{a_1;1}\cdots h_{a_n;n}$ using relations $\mathbf{Q} \cup R_{\leftrightarrow}$, by Lemma 6.2.

Now let R_∇ denote the set of relations

$$e_{ij;a}y_{(k)} = \begin{cases} y_{(i)}h_{a\overline{y};j}e_{ij;1} & \text{if } k = i & (\nabla 1\text{a}) \\ e_{ij;a} & \text{if } k = j & (\nabla 1\text{b}) \\ y_{(k)}e_{ij;a} & \text{otherwise} & (\nabla 1\text{c}) \end{cases}$$

$$y_{(j)}e_{ij;a} = h_{\overline{y}a;j}e_{ij;1} \qquad (\nabla 2)$$

$$y_{(i)}e_{ji;a}e_{ij;b} = h_{\overline{y}ab;i}e_{ij;b}, \qquad (\nabla 3)$$

where $y \in Y$ in each relation, and i,j,k,a,b range over all allowable values, subject to the stated constraints. Note that the assumption that $\overline{y} = y\psi \neq 1$ for all $y \in Y$ (and the assumption that $\mathbb{E}(M) = \{1\} \cup (M \setminus G)$) implies that $a\overline{y}, \overline{y}a \in M \setminus G \subseteq \mathbb{E}(M)$ for all $y \in Y$ and $a \in G$ (so that the words $h_{a\overline{y}}, h_{\overline{y}a}, h_{\overline{y}ab}$ appearing in the above relations are well defined). We aim to prove the following.

Theorem 6.3. *Suppose M is a monoid with group of units G, and that $\mathbb{E}(M) = \langle E(M)\rangle = \{1\} \cup (M \setminus G)$. With the above notation, the idempotent generated subsemigroup $\mathbb{E}(M \wr \mathcal{T}_n) = \langle E(M \wr \mathcal{T}_n)\rangle$ of $M \wr \mathcal{T}_n$ has monoid presentation $\langle \mathbf{Y} \cup X_1 : \mathbf{Q} \cup R_\leftrightarrow \cup R_1' \cup R_\nabla\rangle$ via Θ.*

To prove Theorem 6.3, we first need some preliminary lemmas. We will write $\approx = (\mathbf{Q} \cup R_\leftrightarrow \cup R_1' \cup R_\nabla)^\sharp$ for the congruence on $(\mathbf{Y} \cup X_1)^*$ generated by the relations $\mathbf{Q} \cup R_\leftrightarrow \cup R_1' \cup R_\nabla$. The next result follows by a simple diagrammatic check that the relations R_∇ are preserved by Θ.

Lemma 6.4. *We have* $\approx \subseteq \ker(\Theta)$. □

As usual, proving the reverse containment is more of a challenge.

Lemma 6.5. *If $w \in (\mathbf{Y} \cup X_1)^*$, then $w \approx w_1w_2$ for some $w_1 \in \mathbf{Y}^*$ and $w_2 \in X_1^*$. If $w \notin \mathbf{Y}^*$, then $w_2 \in X_1^+$.*

Proof. For a word $u \in (\mathbf{Y} \cup X_1)^*$, we write $\xi(u)$ for the number of letters from X_1 appearing in u. We prove the lemma by induction on $\xi(w)$. If $\xi(w) = 0$, then we are already done (with $w_1 = w$ and $w_2 = 1$), so suppose $\xi(w) \geq 1$, and write $w = ue_{ij;a}v$, where $u \in (\mathbf{Y} \cup X_1)^*$ and $v \in \mathbf{Y}^*$, so $\xi(u) = \xi(w) - 1$. By (∇1a–∇1c), we have $e_{ij;a}v \approx ze_{ij;b}$ for some $z \in \mathbf{Y}^*$ and some $b \in G$. Since $\xi(uz) = \xi(u) = \xi(w) - 1$, an induction hypothesis gives $uz \approx u_1u_2$ for some $u_1 \in \mathbf{Y}^*$ and $u_2 \in X_1^*$. So $w = ue_{ij;a}v \approx uze_{ij;b} \approx u_1u_2e_{ij;b}$, and we are done (with $w_1 = u_1 \in \mathbf{Y}^*$ and $w_2 = u_2e_{ij;b} \in X_1^+$). (Note that the final assertion in the lemma follows from the above argument.) □

We now improve Lemma 6.5 by showing that the two words w_1, w_2 can be chosen to have a very specific form, in the case that $w \notin \mathbf{Y}^*$.

Lemma 6.6. *Let $w \in (\mathbf{Y} \cup X_1)^* \setminus \mathbf{Y}^*$, and write $w\Theta = (\mathbf{a}, \alpha)$. For $i \in \mathbf{n}$, define*

$$b_i = \begin{cases} 1 & \text{if } a_i \in G \\ a_i & \text{if } a_i \in M \setminus G \end{cases} \qquad \text{and} \qquad c_i = \begin{cases} a_i & \text{if } a_i \in G \\ 1 & \text{if } a_i \in M \setminus G. \end{cases}$$

Then $w \approx w_1w_2$ for some $w_1 \in \mathbf{Y}^*$ and $w_2 \in X_1^+$ with $w_1\Theta = (\mathbf{b}, 1_n)$ and $w_2\Theta = (\mathbf{c}, \alpha)$.

Proof. By Lemma 6.5, the set $\Omega = \{(w_1, w_2) \in \mathbf{Y}^* \times X_1^+ : w \approx w_1w_2\}$ is non-empty. We define a function $\xi : \Omega \to \mathbb{N}$ as follows. Let $(w_1, w_2) \in \Omega$, and write

$$w_1\Theta = (\mathbf{p}, 1_n) \quad \text{and} \quad w_2\Theta = (\mathbf{q}, \alpha) \qquad \text{where } \mathbf{p} \in \mathbb{E}(M)^n \text{ and } \mathbf{q} \in G^n.$$

(Because the last coordinate of $w_1\Theta$ must be 1_n, it follows that the last coordinate of $w_2\Theta$ must be α.) We then define $\xi(w_1, w_2)$ to be the cardinality of the set $\Xi(w_1, w_2) = \{i \in \mathbf{n} : (p_i, q_i) \neq (b_i, c_i)\}$. (Here, we assume that $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$.)

We now choose a pair $(w_1, w_2) \in \Omega$ for which $\xi(w_1, w_2)$ is minimal. We claim that $\xi(w_1, w_2) = 0$. Indeed, suppose to the contrary that $\xi(w_1, w_2) \geq 1$. As above, write $w_1\Theta = (\mathbf{p}, 1_n)$ and $w_2\Theta = (\mathbf{q}, \alpha)$, noting that

$$(\mathbf{a}, \alpha) = w\Theta = (w_1\Theta)(w_2\Theta) = (\mathbf{pq}, \alpha).$$

This gives $p_iq_i = a_i$ for all i. Since $w_2 \in X_1^+$, we have $\alpha \in \mathrm{Sing}_n$, so we may fix some $(i, j) \in \ker(\alpha)$ with $i \neq j$. By relabelling the elements of $\mathbf{n}$, if necessary, we may assume that $(i, j) = (1, 2)$. Define words

$$u_1 = (e_{21;q_1q_2^{-1}}e_{12;q_2}) \cdot (e_{23;q_3}e_{32;1}) \cdots (e_{2n;q_n}e_{n2;1}) \text{ and } u_2 = (e_{12;q_2q_1^{-1}}e_{21;q_1}) \cdot (e_{13;q_3}e_{31;1}) \cdots (e_{1n;q_n}e_{n1;1}),$$

and let v be any word over X (regarded as a subset of X_1 as usual) with $v\Theta = ((1, \ldots, 1), \alpha)$. It is easy to check (diagrammatically) that $u_1\Theta = (\mathbf{q}, \varepsilon_{12})$ and $u_2\Theta = (\mathbf{q}, \varepsilon_{21})$. In particular, since $\alpha = \varepsilon_{12}\alpha = \varepsilon_{21}\alpha$, we have

$$(u_1\Theta)(v\Theta) = (u_2\Theta)(v\Theta) = (\mathbf{q}, \alpha) = w_2\Theta.$$

Since w_2, u_1v, u_2v all belong to X_1^+, Theorem 5.12 then gives $w_2 \approx u_1v \approx u_2v$. As noted earlier, Lemma 6.2 also gives $w_1 \approx h_{p_1;1} \cdots h_{p_n;n}$.

Since $\xi(w_1, w_2) \geq 1$, we may fix some $r \in \Xi(w_1, w_2)$. Note that we could not have $p_r = 1$, or else then $a_r = p_rq_r = q_r \in G$, which would give $(b_r, c_r) = (1, a_r) = (p_r, q_r)$, contradicting our assumption that $r \in \Xi(w_1, w_2)$. In particular, $h_{p_r;r} \neq 1$, so we may write $h_{p_r;r} = (y_1)_{(r)} \cdots (y_k)_{(r)}y_{(r)}$, where $y_1, \ldots, y_k, y \in Y$ (and where p_r is therefore equal to $\overline{y}_1 \cdots \overline{y}_k\overline{y}$). Note that $R_\leftrightarrow$ gives $w_1 \approx w_3h_{p_r;r}$, where $w_3 = h_{p_1;1} \cdots h_{p_{r-1};r-1}h_{p_{r+1};r+1} \cdots h_{p_n;n}$. Note also that

$$p_r \neq 1 \ \Rightarrow\ p_r \in M \setminus G \ \Rightarrow\ a_r = p_rq_r \in M \setminus G \ \Rightarrow\ (b_r, c_r) = (a_r, 1).$$

We now consider separate cases, depending on the value of r.

Case 1. Suppose first that $r \geq 3$. Note that

$$\begin{aligned}
h_{p_r;r}u_1 &= (y_1)_{(r)} \cdots (y_k)_{(r)}y_{(r)}(e_{21;q_1q_2^{-1}}e_{12;q_2}) \cdot (e_{23;q_3}e_{32;1}) \cdots (e_{2,r-1;q_{r-1}}e_{r-1,2;1}) \\
&\qquad\qquad \times (e_{2r;q_r}e_{r2;1})(e_{2,r+1;q_{r+1}}e_{r+1,2;1}) \cdots (e_{2n;q_n}e_{n2;1}) \\
&\approx (y_1)_{(r)} \cdots (y_k)_{(r)}(e_{21;q_1q_2^{-1}}e_{12;q_2}) \cdot (e_{23;q_3}e_{32;1}) \cdots (e_{2,r-1;q_{r-1}}e_{r-1,2;1}) \\
&\qquad\qquad \times y_{(r)}(e_{2r;q_r}e_{r2;1})(e_{2,r+1;q_{r+1}}e_{r+1,2;1}) \cdots (e_{2n;q_n}e_{n2;1}) && \text{by } (\nabla 1\text{c}) \\
&\approx (y_1)_{(r)} \cdots (y_k)_{(r)}(e_{21;q_1q_2^{-1}}e_{12;q_2}) \cdot (e_{23;q_3}e_{32;1}) \cdots (e_{2,r-1;q_{r-1}}e_{r-1,2;1}) \\
&\qquad\qquad \times h_{\overline{y}q_r;r}(e_{2r;1}e_{r2;1})(e_{2,r+1;q_{r+1}}e_{r+1,2;1}) \cdots (e_{2n;q_n}e_{n2;1}) && \text{by } (\nabla 2) \\
&\approx (y_1)_{(r)} \cdots (y_k)_{(r)}h_{\overline{y}q_r;r}(e_{21;q_1q_2^{-1}}e_{12;q_2}) \cdot (e_{23;q_3}e_{32;1}) \cdots (e_{2,r-1;q_{r-1}}e_{r-1,2;1}) \\
&\qquad\qquad \times (e_{2r;1}e_{r2;1})(e_{2,r+1;q_{r+1}}e_{r+1,2;1}) \cdots (e_{2n;q_n}e_{n2;1}) && \text{by } (\nabla 1\text{c}).
\end{aligned}$$

(In the last step of the previous calculation, recall that $h_{\overline{y}q_r;r}$ involves only letters from $Y_{(r)}$.) Note also that

$$((y_1)_{(r)} \cdots (y_k)_{(r)}h_{\overline{y}q_r;r})\Theta = (\overline{y}_1 \cdots \overline{y}_k\overline{y}q_r)_{(r)} = (p_rq_r)_{(r)} = (a_r)_{(r)}.$$

As seen above, $a_r \in M \setminus G$, so Lemma 6.2 gives $(y_1)_{(r)} \cdots (y_k)_{(r)}h_{\overline{y}q_r;r} \approx h_{a_r;r}$. Now put

$$u_3 = (e_{21;q_1q_2^{-1}}e_{12;q_2}) \cdot (e_{23;q_3}e_{32;1}) \cdots (e_{2,r-1;q_{r-1}}e_{r-1,2;1})(e_{2r;1}e_{r2;1})(e_{2,r+1;q_{r+1}}e_{r+1,2;1}) \cdots (e_{2n;q_n}e_{n2;1}).$$

The above calculations show that $h_{p_r;r}u_1 \approx h_{a_r;r}u_3$, and it follows that

$$w \approx w_1w_2 \approx (w_3h_{p_r;r})(u_1v) \approx (w_3h_{a_r;r})(u_3v) = v_1v_2,$$

where $v_1 = w_3h_{a_r;r} \in \mathbf{Y}^*$ and $v_2 = u_3v \in X_1^+$. It follows that $(v_1, v_2) \in \Omega$. But one may easily check that

$$v_1\Theta = ((p_1, \ldots, p_{r-1}, a_r, p_{r+1}, \ldots, p_n), 1_n) \qquad \text{and} \qquad v_2\Theta = ((q_1, \ldots, q_{r-1}, 1, q_{r+1}, \ldots, q_n), \alpha).$$

Since $(b_r, c_r) = (a_r, 1)$, it follows that $\xi(v_1, v_2) = \xi(w_1, w_2) - 1$, contradicting the minimality of $\xi(w_1, w_2)$, and completing the proof of the claim in this case.

Case 2. Next, suppose $r = 1$. So now we have $w_1 \approx w_3h_{p_1;1}$ and $h_{p_1;1} = (y_1)_{(1)} \cdots (y_k)_{(1)}y_{(1)}$. First note that $y_{(1)}e_{21;q_1q_2^{-1}}e_{12;q_2} \approx h_{\overline{y}q_1;1}e_{12;q_2}$, by $(\nabla 3)$. As in the previous case, we have $(y_1)_{(1)} \cdots (y_k)_{(1)}h_{\overline{y}q_1;1} \approx h_{a_1;1}$. It quickly follows that $h_{p_1;1}u_1 \approx h_{a_1;1}u_3$, where $u_3 = e_{12;q_2} \cdot (e_{23;q_3}e_{32;1}) \cdots (e_{2n;q_n}e_{n2;1})$. So $w \approx w_1w_2 \approx (w_3h_{p_1;1})(u_1v) \approx (w_3h_{a_1;1})(u_3v) = v_1v_2$, where $v_1 = w_3h_{a_1;1} \in \mathbf{Y}^*$ and $v_2 = u_3v \in X_1^+$. This time we have

$$v_1\Theta = ((a_1, p_2, \ldots, p_n), 1_n) \qquad \text{and} \qquad v_2\Theta = ((1, q_2, \ldots, q_n), \alpha),$$

and again we obtain $\xi(v_1, v_2) = \xi(w_1, w_2) - 1$, a contradiction.

Case 3. The case in which $r = 2$ is almost identical to the previous case, but we use the word u_2 (defined above) instead of u_1.

This completes the proof of the claim that $\xi(w_1, w_2) = 0$. And this, of course, completes the proof of the lemma. □

We are now ready to tie the loose ends together.

Proof of Theorem 6.3. It remains to prove that $\ker(\Theta) \subseteq \approx$, so suppose $u, v \in (\mathbf{Y} \cup X_1)^*$ are such that $u\Theta = v\Theta$. If $u \in \mathbf{Y}^*$, then $u\Theta \in \mathbb{E}(M) \wr \{1_n\}$, which also gives $v \in \mathbf{Y}^*$; in this case, $u \approx v$ follows from Lemma 6.2. So suppose $u \notin \mathbf{Y}^*$, noting that this also forces $v \notin \mathbf{Y}^*$. Lemma 6.6 then gives $u \approx u_1u_2$ and $v \approx v_1v_2$ for some $u_1, v_1 \in \mathbf{Y}^*$ and $u_2, v_2 \in X_1^+$ with $u_1\Theta = v_1\Theta$ and $u_2\Theta = v_2\Theta$. Lemma 6.2 and Theorem 5.12 (respectively) then give $u_1 \approx v_1$ and $u_2 \approx v_2$. Putting this all together, we obtain $u \approx u_1u_2 \approx v_1v_2 \approx v$. □

Remark 6.7. As noted above, in the case that $M = \mathcal{T}_m$, Theorem 6.3 gives a presentation for the idempotent generated subsemigroup of $\mathcal{T}_m \wr \mathcal{T}_n \cong \mathcal{T}(X, \mathcal{P})$, where $\mathcal{T}(X, \mathcal{P})$ is the endomorphism monoid of a uniform partition $\mathcal{P}$ of the set X into n blocks of size m. Here we have $G = \mathcal{S}_m$, and the monoid presentation $\langle Y : Q\rangle$ for $\mathbb{E}(\mathcal{T}_m) = \{1\} \cup (\mathcal{T}_m \setminus \mathcal{S}_m)$ is deduced from the semigroup presentation for $\mathcal{T}_m \setminus \mathcal{S}_m$ in Theorem 2.4.

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